Completeness of First-Order Logic

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Part I
Logic

1 Syntax of First-Order Languages

1.1 Alphabets

Ebbinghaus–Flum–Thomas, Ch. II · 12/08/2026

Symbols are things one writes down, and a single symbol may itself be a compound of smaller marks.

Definition 1 (Alphabet)**.** An *alphabet* is a nonempty set $\mathbb{A}$ of symbols such that distinct finite sequences of symbols from $\mathbb{A}$, once written out side by side, always yield distinct inscriptions.

Remark 1. $\mathbb{A} = \{\, a_1,\ a_2 \,\}$ with $a_1 := |$ and $a_2 := ||$ is *not* an alphabet: the inscription $|||$ reads as $a_1a_1a_1$, as a_1a_2, and as a_2a_1. Every alphabet below is assumed to satisfy the condition, so an inscription and the sequence of symbols underlying it may be identified without ambiguity. This is what licenses the definitions of string and length that follow.

Definition 2 (Strings, $\mathbb{A}^*$)**.** A *string* (or *word*) over an alphabet $\mathbb{A}$ is a finite sequence of symbols of $\mathbb{A}$. The set of all such is

$$\mathbb{A}^* := \bigcup_{n\in\mathbb{N}} \mathbb{A}^n,$$

a monoid under concatenation, free on $\mathbb{A}$ by unique readability. Its identity is the *empty string* ε, the unique element of $\mathbb{A}^0$.

Definition 3 (Length of a string)**.** The *length* of $\zeta \in \mathbb{A}^*$ is the unique n with $\zeta \in \mathbb{A}^n$, i.e. the number of symbol occurrences in ζ counted with repetition. Denoted by: $|\cdot|$.

Definition 4 (Countable, at most countable)**.** A set M is *countable* if M is not finite and there is a surjection $\alpha : \mathbb{N} \to M$ (i.e. every element of M can be numbered, $M = \{\alpha_n \mid n \in \mathbb{N}\}$). M is *at most countable* if it is finite or countable.

Note the convention: "countable" here excludes the finite case, so at most countable = finite $\vee$ countable, and the two disjuncts are exclusive.

Lemma 1 (E&F II.1.1: at most countable, three equivalent forms)**.** For a nonempty set M the following are equivalent.

(a) M is at most countable.

(b) There is a surjection $\alpha : \mathbb{N} \to M$.

(c) There is an injection $\beta : M \to \mathbb{N}$.

Proof. (a)$\Rightarrow$(b). If M is countable this is the definition. If $M = \{a_0, \dots, a_n\}$ is finite and nonempty, set $\alpha(i) := a_i$ for $i \le n$ and $\alpha(i) := a_0$ otherwise.

(b)$\Rightarrow$(c). Put $\beta(a) := \min\{i \mid \alpha(i) = a\}$, which exists by surjectivity and well-ordering of $\mathbb{N}$. If $\beta(a) = \beta(b) = i$ then $a = \alpha(i) = b$, so β is injective.

(c)⇒(a). Suppose M is not finite; we produce a surjection $\alpha : \mathbb{N} \to M$ by recursion on the ranks of $\beta[M] \subseteq \mathbb{N}$:

$$\alpha(n) := \beta^{-1}\big(\text{the } (n{+}1)\text{-st least element of } \beta[M]\big).$$

This is well defined because β is injective and $\beta[M]$ is infinite, hence unbounded in $\mathbb{N}$, so an $(n{+}1)$-st element always exists. Every $a \in M$ has $\beta(a)$ at some finite rank, so a is in the range of α. □

Lemma 2 (Subsets of at most countable sets)**.** Every subset of an at most countable set is at most countable.

Proof. Let $N \subseteq M$ with M at most countable. If $N = \emptyset$ it is finite. Otherwise $M \neq \emptyset$, so by Lemma 1(c) there is an injection $\beta : M \to \mathbb{N}$; its restriction $\beta \restriction N$ is injective, and Lemma 1 applied to N gives the claim. □

Lemma 3 (Unions of at most countable sets)**.** If M_1, M_2 are at most countable then so is $M_1 \cup M_2$. More generally, if M_n is at most countable for every $n \in \mathbb{N}$, then $\bigcup_{n\in\mathbb{N}} M_n$ is at most countable.

Proof. Binary case: discard empty M_i (trivial otherwise) and take injections $\beta_i : M_i \to \mathbb{N}$. Define $\beta : M_1 \cup M_2 \to \mathbb{N}$ by $\beta(a) := 2\beta_1(a)$ if $a \in M_1$ and $\beta(a) := 2\beta_2(a) + 1$ otherwise. Parity separates the two cases and each half is injective, so β is injective; apply Lemma 1.

Countable case: take injections $\beta_n : M_n \to \mathbb{N}$ (Lemma 1(c)) and let p_n be the n-th prime. Define $\beta : \bigcup_n M_n \to \mathbb{N}$ by

$$\beta(a) := p_{n(a)}^{\beta_{n(a)}(a)+1}, \qquad n(a) := \min\{n \mid a \in M_n\}.$$

The minimum exists since $\mathbb{N}$ is well ordered, so β is well defined. Given $\beta(a)$, unique factorization recovers the unique prime dividing it, hence $n(a)$, hence the exponent and so $\beta_{n(a)}(a)$; since $\beta_{n(a)}$ is injective this recovers a. So β is injective, and Lemma 1 applies. The $+1$ in the exponent is needed: without it every a with $\beta_{n(a)}(a) = 0$ would be sent to 1. □

Remark 2 (Where the choice is)**.** The countable case uses the axiom of countable choice. The hypothesis "each M_n is at most countable" says only that for every n *there exists* an injection $M_n \to \mathbb{N}$, and passing from that to a *sequence* $(\beta_n)_{n\in\mathbb{N}}$ of injections chosen simultaneously is exactly AC_ω. The choice is made in the phrase "take injections β_n", before any coding happens. The book does not mention this choice requirement. The binary case is choice-free: two existential statements can be instantiated one at a time.

Lemma 4 (E&F II.1.2: $\mathbb{A}^*$ is countable)**.** If $\mathbb{A}$ is an at most countable alphabet, then $\mathbb{A}^*$ is countable.

Proof. Enumerate $\mathbb{A}$ without repetition as $a_0, a_1, \dots$ (finite or infinite), and let p_n be the n-th prime. Define $\beta : \mathbb{A}^* \to \mathbb{N}$ by

$$\beta(\varepsilon) := 1, \qquad \beta(a_{i_0} a_{i_1} \cdots a_{i_r}) := p_0^{i_0+1} p_1^{i_1+1} \cdots p_r^{i_r+1}.$$

The exponents recover both the length ($r + 1$ = number of primes used, each exponent being ≥ 1) and each index, so by unique prime factorization β is injective; hence $\mathbb{A}^*$ is at most countable by Lemma 1(c). It is not finite, since $a_0, a_0a_0, a_0a_0a_0, \dots$ are pairwise distinct. Therefore it is countable. □

1.2 The Alphabet of a First-Order Language

Definition 5 (E&F II.2.1: the alphabet $\mathbb{A}_S$ of a first-order language)**.** The alphabet of a first-order language consists of:

(a) variables $v_0, v_1, v_2, \ldots$;

(b) connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$;

(c) quantifiers $\forall, \exists$;

(d) the equality symbol $\equiv$;

(e) parentheses), (;

(f) for each $n \geq 1$ a (possibly empty) set of n-ary relation symbols, for each $n \geq 1$ a (possibly empty) set of n-ary function symbols, and a (possibly empty) set of constants.

Write $\mathbb{A}$ for the set of symbols in (a)–(e), the *logical* symbols, and S for the set of symbols in (f), the *symbol set* (signature); the symbols of S are required to be pairwise distinct and disjoint from $\mathbb{A}$. The alphabet of the first-order language determined by S is

$$\mathbb{A}_S := \mathbb{A} \cup S.$$

1.3 Terms and Formulas

Definition 6 (E&F II.3.1: S-terms, T^S)**.** The set T^S of *S-terms* is the set of strings obtainable by finitely many applications of the following rules (the *calculus of terms*):

(T1) $\dfrac{\ }{x}$ for every variable x;

(T2) $\dfrac{\ }{c}$ for every constant $c \in S$;

(T3) $\dfrac{t_1, \ldots, t_n}{f t_1 \ldots t_n}$ for every n-ary function symbol $f \in S$.

A *derivation* is a finite list of strings each of which is the conclusion of a rule whose premises appear earlier in the list; every entry of a derivation is then itself a term. E.g. for $S = \{f, g, c, R\}$ with f unary and g binary, gv_0fgv_4c is an S-term via

$$c \text{ (T2)}; \quad v_0, v_4 \text{ (T1)}; \quad gv_4c \text{ (T3)}; \quad fgv_4c \text{ (T3)}; \quad gv_0fgv_4c \text{ (T3)}.$$

Definition 7 (E&F II.3.2: S-formulas, L^S)**.** The set L^S of *S-formulas* is the set of strings obtainable by finitely many applications of the following rules (the *calculus of formulas*):

(F1) $t_1 \equiv t_2$, for $t_1, t_2 \in T^S$;

(F2) $Rt_1 \ldots t_n$, for $t_1, \ldots, t_n \in T^S$ and $R \in S$ an n-ary relation symbol;

(F3) $\neg\varphi$, for $\varphi \in L^S$;

(F4) $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$, $(\varphi \leftrightarrow \psi)$, for $\varphi, \psi \in L^S$;

(F5) $\forall x\varphi$ and $\exists x\varphi$, for $\varphi \in L^S$ and x a variable.

L^S is called the *first-order language associated with S*.

Definition 8 (Atomic formulas)**.** The *atomic* S-formulas are exactly those produced by (F1) and (F2), i.e. the strings $t_1 \equiv t_2$ and $Rt_1 \dots t_n$ with $t_i \in T^S$. They are the formulas not built by combining other formulas, hence the base case of every induction on L^S.

Lemma 5 (E&F II.3.3: T^S and L^S are countable)**.** If S is at most countable, then T^S and L^S are countable.

Proof. $\mathbb{A}$ is countable and S is at most countable, so $\mathbb{A}_S = \mathbb{A} \cup S$ is at most countable; by Lemma 4, $\mathbb{A}_S^*$ is countable. Since $T^S, L^S \subseteq \mathbb{A}_S^*$, both are at most countable. Neither is finite: T^S contains all the variables $v_0, v_1, \dots,$ and L^S contains $v_0 \equiv v_0, v_1 \equiv v_1, \dots,$ which are available even when $S = \emptyset$. Hence both are countable. $\square$

1.4 Induction in the Calculi of Terms and of Formulas

Ebbinghaus–Flum–Thomas, Ch. II · 18/08/2026

Write a rule of a calculus $\mathcal{C}$ schematically as $\dfrac{\zeta_1, \dots, \zeta_n}{\zeta}$, allowing $n = 0$ for the premise-free rules. The set Z defined by $\mathcal{C}$ is the smallest set of strings closed under all its rules.

Principle 1 (Induction over a calculus)**.** Let Z be defined by a calculus $\mathcal{C}$ and let P be a property of strings. Suppose that for every rule $\dfrac{\zeta_1, \dots, \zeta_n}{\zeta}$ of $\mathcal{C}$,

$$\zeta_1, \dots, \zeta_n \in Z \text{ all have } P \quad \Longrightarrow \quad \zeta \text{ has } P.$$

Then every element of Z has P. For $n = 0$ the hypothesis is vacuous, so the requirement is that ζ has P outright.

Proof. Define the length of a derivation in $\mathcal{C}$ in the obvious way. The hypothesis gives, by ordinary induction on m, that every string with a derivation of length m has P. Every element of Z has a derivation of some finite length. $\square$

Lemma 6 (E&F II.4.1(a): terms and formulas are nonempty)**.** For every symbol set S, the empty string ε is neither an S-term nor an S-formula.

Proof. Induction on terms with $P(\zeta) :\Longleftrightarrow \zeta \neq \varepsilon$. Variables and constants are nonempty, and $ft_1 \dots t_n$ begins with f, so the induction hypothesis is not even needed. Formulas are the same. $\square$

Unique decomposition

Throughout, $\xi \preceq \eta$ means ξ is an initial segment of η, that is $\eta = \xi\zeta$ for some ζ, and $\prec$ means the segment is proper ($\zeta \neq \varepsilon$). Two facts about strings are used constantly and are immediate from the definition of concatenation.

Lemma 7 (Prefix facts for strings)**.** (i) If $\xi \preceq \eta$ and $\xi' \preceq \eta$ then $\xi \preceq \xi'$ or $\xi' \preceq \xi$.
(ii) Left cancellation: $\xi\zeta = \xi\zeta'$ implies $\zeta = \zeta'$.

Proof. (i) Say $\eta = \xi\zeta = \xi'\zeta'$. If $|\xi| \leq |\xi'|$ then ξ and ξ' agree on their first $|\xi|$ symbols, so $\xi \preceq \xi'$. Otherwise $\xi' \preceq \xi$. (ii) The two strings have the same symbol at each position past $|\xi|$. $\square$

Lemma 8 (E&F II.4.2: no term or formula is a proper initial segment of another)**.** (a) For all $t, t' \in T^S$, $t \not\prec t'$.   (b) For all $\varphi, \varphi' \in L^S$, $\varphi \not\prec \varphi'$.

Proof of (a). Induction on terms for the property

$$P(\eta) :\iff \text{ for every } t' \in T^S,\ t' \not\prec \eta \text{ and } \eta \not\prec t'.$$

$\eta = x$: a term $t' \prec x$ would be the empty string, contradicting Lemma 6. An induction on terms shows x is the only term beginning with the symbol x, so $x \prec t'$ fails as well. The case $\eta = c$ is the same.

$\eta = ft_1 \dots t_n$ with $P(t_1), \dots, P(t_n)$: suppose $t' \prec \eta$, say $\eta = t'\zeta$ with $\zeta \neq \varepsilon$. Then t' begins with f, so t' is neither a variable nor a constant and must come from (T3), say $t' = ft'_1 \dots t'_n$ with the same f and hence the same arity n. Cancelling f,

$$t_1 \dots t_n = t'_1 \dots t'_n \zeta.$$

Both t_1 and t'_1 are initial segments of this common string, so by the prefix facts one is an initial segment of the other, and $P(t_1)$ rules out a proper one. Hence $t_1 = t'_1$, and cancelling it on the left leaves $t_2 \dots t_n = t'_2 \dots t'_n \zeta$. Repeating n times gives $\varepsilon = \zeta$, contradicting $\zeta \neq \varepsilon$. The argument for $\eta \prec t'$ is symmetric. Part (b) is the same induction run over the formula rules. □

Lemma 9 (E&F II.4.3: concatenations of terms match componentwise)**.** If $t_1, \dots, t_n$ and $t'_1, \dots, t'_m$ are S-terms with $t_1 \dots t_n = t'_1 \dots t'_m$, then $n = m$ and $t_i = t'_i$ for all i. Correspondingly for formulas.

Proof. Induction on n. If $n = 0$ the left side is ε, and since every term is nonempty by Lemma 6 this forces $m = 0$. If $n \geq 1$ the left side is nonempty, so $m \geq 1$. Now t_1 and t'_1 are both initial segments of the common string, so one is an initial segment of the other, and Lemma 8(a) forbids a proper one. Hence $t_1 = t'_1$, and left cancellation reduces to $t_2 \dots t_n = t'_2 \dots t'_m$. □

Theorem 1 (E&F II.4.4: unique decomposition of terms and formulas)**.** (a) Every S-term is a variable, a constant, or of the form $ft_1 \dots t_n$ with $f \in S$ n-ary. The three cases are mutually exclusive, and in the third f and $t_1, \dots, t_n$ are uniquely determined.

(b) Every S-formula has one of the forms

$$t_1 \equiv t_2, \quad Rt_1 \dots t_n, \quad \neg\varphi, \quad (\varphi * \psi) \ \ (* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}), \quad \forall x\varphi, \quad \exists x\varphi.$$

The nine cases are mutually exclusive, and in each the constituents displayed are uniquely determined.

Proof. **Existence of a form.** T^S and L^S consist exactly of the strings obtainable by finitely many applications of the rules, so every term and every formula has a derivation. The rule applied at the last step of a derivation of t puts t into one of the three listed forms, and likewise for φ and the nine forms. Nothing more is claimed at this stage, since one string might a priori be produced by several different rules. That is what the rest rules out.

First symbols. A short induction on terms gives

$(*)$ every S-term begins with a variable, a constant, or a function symbol,

since this holds for the rules (T1), (T2) and is inherited by (T3), whose output begins with f. By Definition 5 the symbols of $\mathbb{A}_S$ are pairwise distinct, $\mathbb{A} \cap S = \emptyset$, and each symbol of S carries one fixed arity and one fixed kind. So variables, constants, function symbols, relation symbols, $\neg$, $($, $\forall$ and $\exists$ are eight pairwise disjoint classes of symbols.

(a) Exclusivity. A variable and a constant are single symbols lying in disjoint classes, and $ft_1 \dots t_n$ begins with a function symbol. So the three cases have first symbols in disjoint classes and cannot coincide.

(a) Uniqueness. Suppose $ft_1 \dots t_n = f't'_1 \dots t'_m$. The first symbols give $f = f'$, and since arity is a function of the symbol, $n = m$. Left cancellation of f gives $t_1 \dots t_n = t'_1 \dots t'_n$, and Lemma 9 gives $t_i = t'_i$ for every i.

(b) Exclusivity. Read off the first symbol of each form. By $(*)$, $t_1 \equiv t_2$ begins with a variable, a constant or a function symbol. $Rt_1 \dots t_n$ begins with a relation symbol, $\neg\varphi$ with $\neg$, each $(\varphi * \psi)$ with (, and $\forall x\varphi$, $\exists x\varphi$ with $\forall$, $\exists$. These lie in disjoint classes, which separates the six groups. It does not separate the four cases inside $(\varphi * \psi)$, since all four begin with (, and that is settled by the uniqueness argument below, which shows the connective is determined by the string.

(b) Uniqueness, case by case.

Equations. Let $t_1 \equiv t_2 = t'_1 \equiv t'_2$. Both t_1 and t'_1 are initial segments of this string, so one is an initial segment of the other, and Lemma 8(a) forces $t_1 = t'_1$. Cancelling t_1 and then the symbol $\equiv$ leaves $t_2 = t'_2$.

Relational atoms. Let $Rt_1 \dots t_n = R't'_1 \dots t'_m$. First symbols give $R = R'$, hence $n = m$ by fixed arity. Cancel R and apply Lemma 9.

Negations. $\neg\varphi = \neg\varphi'$ gives $\varphi = \varphi'$ by cancelling $\neg$.

Binary connectives. Let $(\varphi * \psi) = (\varphi' *' \psi')$. Cancel the opening parenthesis to get

$$\varphi * \psi) \;=\; \varphi' *' \psi').$$

Both φ and φ' are initial segments of this string, so one is an initial segment of the other, and Lemma 8(b) forces $\varphi = \varphi'$. Cancelling φ leaves $*\psi) = *'\psi')$, whose first symbols give $* = *'$. This is the step that separates cases (4) through (7). Cancelling $*$ leaves $\psi) = \psi')$, and Lemma 8(b) applied to the initial segments ψ and ψ' gives $\psi = \psi'$.

Quantifications. $\forall x\varphi = \forall x'\varphi'$ gives $x = x'$ by comparing the second symbol, since a variable is a single symbol, and then $\varphi = \varphi'$ by cancellation. Same for $\exists$, and $\forall$ against $\exists$ is excluded by the first symbol. □

Remark 3. The two loads are carried by Lemma 8 and by the disjointness of the symbol classes. Without Lemma 8 there is no way to match φ against φ' inside $(\varphi * \psi)$, because knowing that both are formulas and both are initial segments of the same string is worthless if one may be a proper initial segment of the other.

Prefix notation is what makes Lemma 8 work, and the delimiting parentheses in (F4) do the same job for formulas. Drop them and Theorem 1 fails: with $\chi := \exists v_0 P v_0 \wedge Q v_1$ one gets two readings, so any function defined by recursion on formulas is no longer well defined (E&F 4.7).

Principle 2 (Definition by recursion on terms and formulas)**.** By Theorem 1(a), a function on T^S is well defined by assigning a value to each variable, a value to each constant, and a value to $ft_1 \dots t_n$ in terms of the values already assigned to $t_1, \dots, t_n$. Correspondingly for L^S by Theorem 1(b).

Remark 4. This is the point of the section. Unique decomposition is what licenses recursion, and every syntactic function defined later (the satisfaction relation, substitution, Gödel numbering) is given this way. The pattern is not specific to first-order logic. Any syntax presented by a grammar or a calculus needs the same theorem before recursion on it is legitimate.

Definition 9 (E&F II.4.5: var(t) and SF(φ))**.** var(t), the set of variables occurring in an S-term t:

$$\begin{aligned} \mathrm{var}(x) &:= \{x\}, \\ \mathrm{var}(c) &:= \emptyset, \\ \mathrm{var}(ft_1 \dots t_n) &:= \mathrm{var}(t_1) \cup \dots \cup \mathrm{var}(t_n). \end{aligned}$$

SF(φ), the set of subformulas of φ:

$$\begin{aligned} \mathrm{SF}(t_1 \equiv t_2) &:= \{t_1 \equiv t_2\}, \\ \mathrm{SF}(Rt_1 \dots t_n) &:= \{Rt_1 \dots t_n\}, \\ \mathrm{SF}(\neg\varphi) &:= \{\neg\varphi\} \cup \mathrm{SF}(\varphi), \\ \mathrm{SF}((\varphi * \psi)) &:= \{(\varphi * \psi)\} \cup \mathrm{SF}(\varphi) \cup \mathrm{SF}(\psi) \quad (* = \wedge, \vee, \rightarrow, \leftrightarrow), \\ \mathrm{SF}(\forall x\varphi) &:= \{\forall x\varphi\} \cup \mathrm{SF}(\varphi), \\ \mathrm{SF}(\exists x\varphi) &:= \{\exists x\varphi\} \cup \mathrm{SF}(\varphi). \end{aligned}$$

1.5 Free Variables and Sentences

An occurrence of a variable is *bound* if it lies in the scope of a quantifier on that variable, and *free* otherwise. The same variable can have both kinds of occurrence in one formula, as y does in $\exists x(Ryz \wedge \forall y(\neg y \equiv x \vee Ryz))$.

Definition 10 (E&F II.5.1: free(φ))**.** free(φ), by recursion on formulas:

$$\begin{aligned} \mathrm{free}(t_1 \equiv t_2) &:= \mathrm{var}(t_1) \cup \mathrm{var}(t_2), \\ \mathrm{free}(Rt_1 \dots t_n) &:= \mathrm{var}(t_1) \cup \dots \cup \mathrm{var}(t_n), \\ \mathrm{free}(\neg\varphi) &:= \mathrm{free}(\varphi), \\ \mathrm{free}((\varphi * \psi)) &:= \mathrm{free}(\varphi) \cup \mathrm{free}(\psi) \quad (* = \wedge, \vee, \rightarrow, \leftrightarrow), \\ \mathrm{free}(\forall x\varphi) &:= \mathrm{free}(\varphi) \setminus \{x\}, \\ \mathrm{free}(\exists x\varphi) &:= \mathrm{free}(\varphi) \setminus \{x\}. \end{aligned}$$

An S-formula φ with free(φ) $= \emptyset$ is an *S-sentence*. Write

$$L_n^S := \{\varphi \in L^S \mid \mathrm{free}(\varphi) \subseteq \{v_0, \dots, v_{n-1}\}\},$$

so L_0^S is the set of S-sentences.

2 Semantics of First-Order Languages

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Unlike the previous chapter everything here depends on what the strings are taken to mean, so the concepts are *semantic* rather than *syntactic*.

2.1 Structures and Interpretations

Definition 11 (E&F III.1.1: S-structure $\mathfrak{A} = (A, \mathfrak{a})$)**.** An *$S$-structure* is a pair $\mathfrak{A} = (A, \mathfrak{a})$ where A is a nonempty set, the *domain* or *universe*, and $\mathfrak{a}$ is a map on S with

(1) $\mathfrak{a}(R)$ an n-ary relation on A, for $R \in S$ n-ary relational,

(2) $\mathfrak{a}(f)$ an n-ary function on A, for $f \in S$ n-ary functional,

(3) $\mathfrak{a}(c) \in A$, for $c \in S$ a constant.

Nonemptiness of A is a convention, and it is what makes $\exists x\, x \equiv x$ valid. Write $R^{\mathfrak{A}}, f^{\mathfrak{A}}, c^{\mathfrak{A}}$ for $\mathfrak{a}(R), \mathfrak{a}(f), \mathfrak{a}(c)$.

Definition 12 (E&F III.1.2: assignment β)**.** An *assignment* in an S-structure $\mathfrak{A}$ is a map $\beta : \{v_n \mid n \in \mathbb{N}\} \to A$.

Definition 13 (E&F III.1.3: S-interpretation $\mathfrak{I} = (\mathfrak{A}, \beta)$)**.** An *$S$-interpretation* is a pair $\mathfrak{I} = (\mathfrak{A}, \beta)$ with $\mathfrak{A}$ an S-structure and β an assignment in $\mathfrak{A}$.

The structure interprets S, the assignment interprets the variables. For $a \in A$ and a variable x, let $\beta\frac{a}{x}$ be the assignment agreeing with β off x and sending x to a,

$$\beta\frac{a}{x}(y) := \begin{cases} \beta(y) & y \neq x, \\ a & y = x, \end{cases} \qquad \mathfrak{I}\frac{a}{x} := \big(\mathfrak{A}, \beta\frac{a}{x}\big).$$

A connective is *extensional* if the truth value of a compound depends only on the truth values of its constituents. And all the connectives in the metalanguage are extensional and defined as expected.

2.2 The Satisfaction Relation

Definition 14 (E&F III.3.1: value $\mathfrak{I}(t)$ of a term)**.** For $\mathfrak{I} = (\mathfrak{A}, \beta)$, define $\mathfrak{I}(t) \in A$ by recursion on terms:

$$\mathfrak{I}(x) := \beta(x), \qquad \mathfrak{I}(c) := c^{\mathfrak{A}}, \qquad \mathfrak{I}(ft_1 \dots t_n) := f^{\mathfrak{A}}\big(\mathfrak{I}(t_1), \dots, \mathfrak{I}(t_n)\big).$$

Definition 15 (E&F III.3.2: the satisfaction relation $\mathfrak{I} \models \varphi$)**.** Define $\mathfrak{I} \models \varphi$ ($\mathfrak{I}$ is a *model* of φ, or φ *holds* in $\mathfrak{I}$) by recursion on formulas:

$$\begin{aligned}
\mathfrak{I} \models t_1 \equiv t_2 \ &:\Longleftrightarrow\ \mathfrak{I}(t_1) = \mathfrak{I}(t_2),\\
\mathfrak{I} \models Rt_1 \dots t_n \ &:\Longleftrightarrow\ R^{\mathfrak{A}}\mathfrak{I}(t_1) \dots \mathfrak{I}(t_n),\\
\mathfrak{I} \models \neg\varphi \ &:\Longleftrightarrow\ \text{not } \mathfrak{I} \models \varphi,\\
\mathfrak{I} \models (\varphi \wedge \psi) \ &:\Longleftrightarrow\ \mathfrak{I} \models \varphi \text{ and } \mathfrak{I} \models \psi,\\
\mathfrak{I} \models (\varphi \vee \psi) \ &:\Longleftrightarrow\ \mathfrak{I} \models \varphi \text{ or } \mathfrak{I} \models \psi,\\
\mathfrak{I} \models (\varphi \to \psi) \ &:\Longleftrightarrow\ \text{if } \mathfrak{I} \models \varphi \text{ then } \mathfrak{I} \models \psi,\\
\mathfrak{I} \models (\varphi \leftrightarrow \psi) \ &:\Longleftrightarrow\ \mathfrak{I} \models \varphi \text{ iff } \mathfrak{I} \models \psi,\\
\mathfrak{I} \models \forall x\varphi \ &:\Longleftrightarrow\ \text{for all } a \in A,\ \mathfrak{I}\frac{a}{x} \models \varphi,\\
\mathfrak{I} \models \exists x\varphi \ &:\Longleftrightarrow\ \text{for some } a \in A,\ \mathfrak{I}\frac{a}{x} \models \varphi.
\end{aligned}$$

For a set Φ of formulas, $\mathfrak{I} \models \Phi$ means $\mathfrak{I} \models \varphi$ for every $\varphi \in \Phi$.

This is legitimate only because of Theorem 1(b). The metalanguage connectives on the right are used extensionally in the sense fixed above, so the clauses agree with the truth tables.

2.3 The Consequence Relation

Definition 16 (E&F III.4.1: the consequence relation $\Phi \models \varphi$)**.** For a set of formulas Φ and a formula φ, say φ is a *consequence* of Φ, written $\Phi \models \varphi$, iff every interpretation which is a model of Φ is a model of φ. Write $\psi \models \varphi$ for $\{\psi\} \models \varphi$.

The symbol $\models$ carries both meanings, disambiguated by whether an interpretation or a set of formulas stands to its left.

Definition 17 (E&F III.4.2, 4.3: valid, satisfiable)**.** φ is *valid*, written $\models \varphi$, iff $\emptyset \models \varphi$, that is, iff φ holds under every interpretation. φ is *satisfiable*, written $\mathrm{Sat}\,\varphi$, iff some interpretation is a model of it, and $\mathrm{Sat}\,\Phi$ iff some interpretation is a model of every formula in Φ.

Lemma 10 (E&F III.4.4: consequence expressed by unsatisfiability)**.** $\Phi \models \varphi$ iff not $\mathrm{Sat}\,\Phi \cup \{\neg\varphi\}$. In particular φ is valid iff $\neg\varphi$ is not satisfiable.

Proof. $\Phi \models \varphi$
iff every interpretation which is a model of Φ is a model of φ,
iff there is no interpretation which is a model of Φ but not of φ,
iff there is no interpretation which is a model of Φ and of $\neg\varphi$,
iff there is no interpretation which is a model of $\Phi \cup \{\neg\varphi\}$, □

Remark 5. Failure of $\Phi \models \varphi$ does not give $\Phi \models \neg\varphi$.

Definition 18 (E&F III.4.5: logical equivalence $=\!\!\!|\!\models$)**.** φ and ψ are *logically equivalent*, written $\varphi =\!\!\!|\!\models \psi$, iff $\varphi \models \psi$ and $\psi \models \varphi$, equivalently iff $\models \varphi \leftrightarrow \psi$.

Reduction of the connectives

From the satisfaction clauses and the truth tables,

$$(\varphi \wedge \psi) =\!\!\!|\!\models \neg(\neg\varphi \vee \neg\psi), \quad (\varphi \to \psi) =\!\!\!|\!\models \neg\varphi \vee \psi, \quad \forall x\varphi =\!\!\!|\!\models \neg\exists x\neg\varphi,$$

and similarly for $\leftrightarrow$. So a map * defined by recursion on formulas sends each φ to a logically equivalent φ^* free of $\wedge, \to, \leftrightarrow, \forall$.

Remark 6 (Convention from here on)**.** The alphabet keeps only $\neg, \vee, \exists$, rule (F4) is restricted to $(\varphi \vee \psi)$ and (F5) to $\exists x\varphi$, and the corresponding satisfaction clauses are dropped. The other symbols remain in use as abbreviations for their *-translations. Inductions on formulas then have three cases instead of nine, which is the point.

Lemma 11 (E&F III.4.6: Coincidence Lemma)**.** Let $\mathfrak{I}_1 = (\mathfrak{A}_1, \beta_1)$ be an S_1-interpretation and $\mathfrak{I}_2 = (\mathfrak{A}_2, \beta_2)$ an S_2-interpretation with $A_1 = A_2$, and put $S := S_1 \cap S_2$.

(a) If t is an S-term and the two agree on the S-symbols and the variables occurring in t, then $\mathfrak{I}_1(t) = \mathfrak{I}_2(t)$.

(b) If φ is an S-formula and the two agree on the S-symbols occurring in φ and on $\mathrm{free}(\varphi)$, then $\mathfrak{I}_1 \models \varphi$ iff $\mathfrak{I}_2 \models \varphi$.

Proof. Call a pair $(\mathfrak{I}_1, \mathfrak{I}_2)$ *admissible* if it is as in the statement, that is $\mathfrak{I}_1$ is an S_1-interpretation, $\mathfrak{I}_2$ is an S_2-interpretation, and $A_1 = A_2 =: A$. Say the pair *agrees on* a symbol $k \in S$ iff $k^{\mathfrak{A}_1} = k^{\mathfrak{A}_2}$, and agrees on a variable y iff $\beta_1(y) = \beta_2(y)$.

proof of (a) Fix an admissible pair $(\mathfrak{I}_1, \mathfrak{I}_2)$. This is legitimate here because the term clauses never modify the interpretation. Induct on terms with the property

$$P(t) \;:\Longleftrightarrow\; \text{if the pair agrees on every } S\text{-symbol occurring in } t \text{ and on every } y \in \text{var}(t), \text{ then } \mathfrak{I}_1(t) = \mathfrak{I}_2(t).$$

Note P is a conditional, so each case assumes its own antecedent.

(T1) $t = x$. Assume the antecedent. Since $\text{var}(x) = \{x\}$ it gives $\beta_1(x) = \beta_2(x)$, so $\mathfrak{I}_1(x) = \beta_1(x) = \beta_2(x) = \mathfrak{I}_2(x)$.

(T2) $t = c$. Assume the antecedent. The only S-symbol occurring in c is c itself, so $c^{\mathfrak{A}_1} = c^{\mathfrak{A}_2}$, and $\mathfrak{I}_1(c) = c^{\mathfrak{A}_1} = c^{\mathfrak{A}_2} = \mathfrak{I}_2(c)$.

(T3) $t = ft_1 \dots t_n$, induction hypothesis $P(t_1), \dots, P(t_n)$. Assume the antecedent for t. Every S-symbol occurring in t_i occurs in t, and $\text{var}(t_i) \subseteq \text{var}(t)$, so the antecedent of $P(t_i)$ holds for the same pair. Hence $\mathfrak{I}_1(t_i) = \mathfrak{I}_2(t_i)$ for each i. Also f occurs in t, so $f^{\mathfrak{A}_1} = f^{\mathfrak{A}_2}$. Therefore

$$\begin{aligned}
\mathfrak{I}_1(ft_1 \dots t_n) &= f^{\mathfrak{A}_1}\big(\mathfrak{I}_1(t_1), \dots, \mathfrak{I}_1(t_n)\big) \\
&= f^{\mathfrak{A}_1}\big(\mathfrak{I}_2(t_1), \dots, \mathfrak{I}_2(t_n)\big) && \text{(induction hypothesis)} \\
&= f^{\mathfrak{A}_2}\big(\mathfrak{I}_2(t_1), \dots, \mathfrak{I}_2(t_n)\big) && (f^{\mathfrak{A}_1} = f^{\mathfrak{A}_2}) \\
&= \mathfrak{I}_2(ft_1 \dots t_n).
\end{aligned}$$

Since (a) is now proved for *every* admissible pair, it may be invoked at any pair, which is what (b) needs.

Proof of (b) Here the pair may *not* be fixed in advance. The clause for $\exists$ applies the induction hypothesis to $\mathfrak{I}_1\frac{a}{x}$ and $\mathfrak{I}_2\frac{a}{x}$, which form a different pair, so the property carried through the induction must be quantified over pairs:

$$\begin{aligned}
Q(\varphi) \;:\Longleftrightarrow\;& \text{for } \textit{every} \text{ admissible pair } (\mathfrak{I}_1, \mathfrak{I}_2): \\
& \quad \text{if the pair agrees on every } S\text{-symbol occurring in } \varphi \\
& \quad \text{and on every } y \in \text{free}(\varphi), \\
& \quad \text{then } \big(\mathfrak{I}_1 \models \varphi \iff \mathfrak{I}_2 \models \varphi\big).
\end{aligned}$$

(F1) $\varphi = t_1 \equiv t_2$. Take any admissible pair satisfying the antecedent. Here $\text{free}(\varphi) = \text{var}(t_1) \cup \text{var}(t_2)$ and the S-symbols of t_1 and of t_2 are among those of φ, so the hypothesis of (a) holds for t_1 and for t_2 at this pair. Thus $\mathfrak{I}_1(t_j) = \mathfrak{I}_2(t_j)$ for $j = 1, 2$, and

$$\mathfrak{I}_1 \models t_1 \equiv t_2 \iff \mathfrak{I}_1(t_1) = \mathfrak{I}_1(t_2) \iff \mathfrak{I}_2(t_1) = \mathfrak{I}_2(t_2) \iff \mathfrak{I}_2 \models t_1 \equiv t_2.$$

(F2) $\varphi = Rt_1 \dots t_n$. As above, $\mathfrak{I}_1(t_i) = \mathfrak{I}_2(t_i)$ by (a), and R occurs in φ so $R^{\mathfrak{A}_1} = R^{\mathfrak{A}_2}$. Then

$$\begin{aligned}
\mathfrak{I}_1 \models Rt_1 \dots t_n &\iff R^{\mathfrak{A}_1}\mathfrak{I}_1(t_1) \dots \mathfrak{I}_1(t_n) \\
&\iff R^{\mathfrak{A}_1}\mathfrak{I}_2(t_1) \dots \mathfrak{I}_2(t_n) && \text{(by (a))} \\
&\iff R^{\mathfrak{A}_2}\mathfrak{I}_2(t_1) \dots \mathfrak{I}_2(t_n) && (R^{\mathfrak{A}_1} = R^{\mathfrak{A}_2}) \\
&\iff \mathfrak{I}_2 \models Rt_1 \dots t_n.
\end{aligned}$$

(F3) $\varphi = \neg\psi$, induction hypothesis $Q(\psi)$. The S-symbols of $\neg\psi$ are exactly those of ψ, and $\text{free}(\neg\psi) = \text{free}(\psi)$. So the antecedent for $\neg\psi$ at a given pair is literally the antecedent for ψ at that same pair, and $Q(\psi)$ applies there. Hence

$$\mathfrak{I}_1 \models \neg\psi \iff \text{not } \mathfrak{I}_1 \models \psi \iff \text{not } \mathfrak{I}_2 \models \psi \iff \mathfrak{I}_2 \models \neg\psi.$$

(F4) $\varphi = (\psi \vee \chi)$, induction hypotheses $Q(\psi), Q(\chi)$. The S-symbols of ψ and of χ are among those of φ, and $\text{free}(\psi), \text{free}(\chi) \subseteq \text{free}(\varphi)$. So the antecedent for φ at a pair implies the antecedent for each of ψ, χ at that same pair, and

$$\mathfrak{I}_1 \models (\psi \vee \chi) \iff \mathfrak{I}_1 \models \psi \text{ or } \mathfrak{I}_1 \models \chi \iff \mathfrak{I}_2 \models \psi \text{ or } \mathfrak{I}_2 \models \chi \iff \mathfrak{I}_2 \models (\psi \vee \chi).$$

(F5) $\varphi = \exists x\psi$, induction hypothesis $Q(\psi)$. Take an admissible pair $(\mathfrak{I}_1, \mathfrak{I}_2)$ satisfying the antecedent for $\exists x\psi$, so it agrees on the S-symbols occurring in $\exists x\psi$ and on $\text{free}(\exists x\psi) = \text{free}(\psi) \setminus \{x\}$. Fix $a \in A$ and consider the pair $\left(\mathfrak{I}_1 \frac{a}{x}, \mathfrak{I}_2 \frac{a}{x}\right)$. It is admissible, since updating an assignment changes neither structure nor domain.

Claim: it satisfies the antecedent for ψ.

- *Symbols.* ψ and $\exists x\psi$ contain the same S-symbols, and $\frac{a}{x}$ leaves $\mathfrak{A}_1, \mathfrak{A}_2$ untouched, so agreement on symbols transfers unchanged.
- *Variables.* Let $y \in \text{free}(\psi)$. If $y = x$ then $\beta_1 \frac{a}{x}(x) = a = \beta_2 \frac{a}{x}(x)$. If $y \neq x$ then $y \in \text{free}(\psi) \setminus \{x\} = \text{free}(\exists x\psi)$, so $\beta_1(y) = \beta_2(y)$ by the antecedent, and $\beta_i \frac{a}{x}(y) = \beta_i(y)$ for $i = 1, 2$, giving $\beta_1 \frac{a}{x}(y) = \beta_2 \frac{a}{x}(y)$.

This case split is the whole content of the step.

By the claim and $Q(\psi)$,

$$\mathfrak{I}_1 \frac{a}{x} \models \psi \iff \mathfrak{I}_2 \frac{a}{x} \models \psi \qquad \text{for every } a \in A.$$

Therefore

$$\begin{aligned} \mathfrak{I}_1 \models \exists x\psi &\iff \text{there is } a \in A_1 \text{ with } \mathfrak{I}_1 \tfrac{a}{x} \models \psi \\ &\iff \text{there is } a \in A_2 \text{ with } \mathfrak{I}_2 \tfrac{a}{x} \models \psi && (A_1 = A_2) \\ &\iff \mathfrak{I}_2 \models \exists x\psi. && \square \end{aligned}$$

Dropping the assignment

An assignment β fixes a value for every one of the infinitely many variables. The Coincidence Lemma says that almost all of this is irrelevant. Whether $\mathfrak{I} \models \varphi$ depends only on the values β gives to the variables in $\text{free}(\varphi)$, and that set is finite, since φ is a finite string. So a finite list of elements carries all the information about β that matters.

This allows for a lighter notation. Let $\varphi \in L_n^S$, so that $\text{free}(\varphi) \subseteq \{v_0, \ldots, v_{n-1}\}$, and let $a_0, \ldots, a_{n-1} \in A$. Write

$$\mathfrak{A} \models \varphi[a_0, \ldots, a_{n-1}]$$

to mean $(\mathfrak{A}, \beta) \models \varphi$ for some, equivalently every, assignment β with $\beta(v_i) = a_i$ for $i < n$. Similarly write $t^{\mathfrak{A}}[a_0, \ldots, a_{n-1}]$ for $\mathfrak{I}(t)$ when $\text{var}(t) \subseteq \{v_0, \ldots, v_{n-1}\}$.

If φ is a sentence we may take $n = 0$. The list is then empty and we write $\mathfrak{A} \models \varphi$, with no assignment mentioned at all.

Dropping the symbol set

Definition 19 (E&F III.4.7: reduct $\mathfrak{A}'|_S$ and expansion)**.** Let $S \subseteq S'$, let $\mathfrak{A} = (A, \mathfrak{a})$ be an S-structure and let $\mathfrak{A}' = (A', \mathfrak{a}')$ be an S'-structure. Then $\mathfrak{A}$ is the *S-reduct* of $\mathfrak{A}'$, written $\mathfrak{A} = \mathfrak{A}'|_S$, iff

(a) $A = A'$, and

(b) $\mathfrak{a}$ and $\mathfrak{a}'$ agree on S, that is $k^{\mathfrak{A}} = k^{\mathfrak{A}'}$ for every $k \in S$.

In this situation $\mathfrak{A}'$ is called an *expansion* of $\mathfrak{A}$.

So a reduct forgets the interpretation of the symbols in $S' \setminus S$ and changes nothing else. For example $\mathfrak{R} = \mathfrak{R}^{<}|_{S_{\text{ar}}}$, the ordered field of reals forgetting its ordering.

A structure and its reduct interpret every symbol of S alike and have the same domain, so the Coincidence Lemma applies to the pair. Hence for every $\varphi \in L_n^S$,

$$\mathfrak{A}'|_S \models \varphi[a_0, \dots, a_{n-1}] \quad \text{iff} \quad \mathfrak{A}' \models \varphi[a_0, \dots, a_{n-1}].$$

Lemma 12 (E&F III.4.8: satisfiability does not depend on the symbol set)**.** Let $\Phi \subseteq L^S$ and $S \subseteq S'$. Then Φ is satisfiable when read as a set of S-formulas iff it is satisfiable when read as a set of S'-formulas.

Proof. Suppose an S'-interpretation $(\mathfrak{A}', \beta)$ models Φ. Then the S-interpretation $(\mathfrak{A}'|_S, \beta)$ also models Φ, by the displayed equivalence above.

Conversely, suppose an S-interpretation $(\mathfrak{A}, \beta)$ models Φ. Expand $\mathfrak{A}$ to an S'-structure $\mathfrak{A}'$ by interpreting the symbols of $S' \setminus S$ arbitrarily, which is possible since A is nonempty. Then $\mathfrak{A}'|_S = \mathfrak{A}$, so $(\mathfrak{A}', \beta)$ models Φ. □

Satisfiability therefore needs no reference to a symbol set, and from here on we write $\operatorname{Sat}\Phi$ without saying over which S. The same argument works for the consequence relation.

2.4 Two Lemmas on the Satisfaction Relation

Definition 20 (E&F III.5.1: isomorphism $\pi : \mathfrak{A} \cong \mathfrak{B}$)**.** For S-structures $\mathfrak{A}, \mathfrak{B}$, a map $\pi : A \to B$ is an *isomorphism*, written $\pi : \mathfrak{A} \cong \mathfrak{B}$, iff π is a bijection and for all $a_1, \dots, a_n \in A$,

$$R^{\mathfrak{A}} a_1 \dots a_n \iff R^{\mathfrak{B}} \pi(a_1) \dots \pi(a_n), \quad \pi\big(f^{\mathfrak{A}}(a_1, \dots, a_n)\big) = f^{\mathfrak{B}}(\pi(a_1), \dots, \pi(a_n)), \quad \pi(c^{\mathfrak{A}}) = c^{\mathfrak{B}}.$$

$\mathfrak{A} \cong \mathfrak{B}$ iff such a π exists.

Lemma 13 (E&F III.5.2: Isomorphism Lemma)**.** If $\mathfrak{A} \cong \mathfrak{B}$ then for every S-sentence φ, $\mathfrak{A} \models \varphi$ iff $\mathfrak{B} \models \varphi$.

Proof. The lemma is about sentences, but sentences cannot carry an induction on formulas. A subformula of a sentence usually has free variables, as ψ does inside the sentence $\exists x \psi$. So we prove a statement about all formulas and recover the lemma at the end.

Once formulas are allowed, assignments enter, and here they live in different places: an assignment for $\mathfrak{A}$ takes values in A, one for $\mathfrak{B}$ takes values in B. The isomorphism supplies the obvious translation. Given $\pi : \mathfrak{A} \cong \mathfrak{B}$ and an assignment β in $\mathfrak{A}$, set

$$\beta^\pi := \pi \circ \beta, \qquad \mathfrak{I}^\pi := (\mathfrak{B}, \beta^\pi) \quad \text{where } \mathfrak{I} = (\mathfrak{A}, \beta).$$

So $\mathfrak{I}^\pi$ is $\mathfrak{I}$ with everything carried across π. The proof is two claims.

Claim 1: $\pi(\mathfrak{I}(t)) = \mathfrak{I}^\pi(t)$ for every term t. Evaluating a term in $\mathfrak{A}$ and then applying π gives the same element as applying π first and evaluating in $\mathfrak{B}$. Induction on terms. For a variable this is the definition of β^π. For a constant it is the clause $\pi(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$. For $ft_1 \dots t_n$ it is the clause $\pi(f^{\mathfrak{A}}(\bar{a})) = f^{\mathfrak{B}}(\pi\bar{a})$ applied to the induction hypothesis.

Claim 2: $\mathfrak{I} \models \varphi$ iff $\mathfrak{I}^\pi \models \varphi$, for every formula φ and every assignment β in $\mathfrak{A}$. As in Lemma 11, the claim is quantified over all β, because the quantifier case will apply the induction hypothesis at a modified assignment.

The atomic and $\neg, \vee$ cases are routine, and it is worth noting only which part of Definition 20 each one consumes.

- $t_1 \equiv t_2$ uses *injectivity*: $\mathfrak{I}(t_1) = \mathfrak{I}(t_2)$ iff $\pi(\mathfrak{I}(t_1)) = \pi(\mathfrak{I}(t_2))$, then Claim 1.
- $Rt_1 \dots t_n$ uses the *relation clause*, then Claim 1.
- $\neg$ and $\vee$ use only the induction hypothesis.

The quantifier case $\varphi = \exists x \psi$. This is the only step with content. It rests on the identity

$$\left(\mathfrak{I}\tfrac{a}{x}\right)^\pi = \mathfrak{I}^\pi \tfrac{\pi(a)}{x} \qquad (a \in A),$$

which says that changing x to a and then transporting is the same as transporting and then changing x to $\pi(a)$. To see it, compare the two assignments at each variable: at x both give $\pi(a)$, and at $y \neq x$ both give $\pi(\beta(y))$.

Now read off the chain, using the induction hypothesis at the second step and the identity at the third:

$$\begin{aligned}
\mathfrak{I} \models \exists x \psi &\iff \text{there is } a \in A \text{ with } \mathfrak{I}\tfrac{a}{x} \models \psi \\
&\iff \text{there is } a \in A \text{ with } \left(\mathfrak{I}\tfrac{a}{x}\right)^\pi \models \psi \\
&\iff \text{there is } a \in A \text{ with } \mathfrak{I}^\pi \tfrac{\pi(a)}{x} \models \psi \\
&\iff \text{there is } b \in B \text{ with } \mathfrak{I}^\pi \tfrac{b}{x} \models \psi \qquad\qquad (\pi \text{ surjective}) \\
&\iff \mathfrak{I}^\pi \models \exists x \psi.
\end{aligned}$$

The fourth step is where *surjectivity* is needed. Without it the elements of B outside the range of π would never be tested, and the right side could hold while the left fails.

Finally let φ be a sentence. Claim 2 gives $(\mathfrak{A}, \beta) \models \varphi$ iff $(\mathfrak{B}, \beta^\pi) \models \varphi$ for any β, and since $\mathrm{free}(\varphi) = \emptyset$ neither side depends on its assignment. So $\mathfrak{A} \models \varphi$ iff $\mathfrak{B} \models \varphi$. □

Corollary 1 (E&F III.5.3: isomorphism with parameters)**.** If $\pi : \mathfrak{A} \cong \mathfrak{B}$ then for $\varphi \in L_n^S$, $\mathfrak{A} \models \varphi[a_0, \dots, a_{n-1}]$ iff $\mathfrak{B} \models \varphi[\pi(a_0), \dots, \pi(a_{n-1})]$.

The converse fails. Structures satisfying the same sentences need not be isomorphic (we will see this later).

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Definition 21 (E&F III.5.4: substructure $\mathfrak{A} \subseteq \mathfrak{B}$)**.** Let $\mathfrak{A}, \mathfrak{B}$ be S-structures. Then $\mathfrak{A}$ is a *substructure* of $\mathfrak{B}$, written $\mathfrak{A} \subseteq \mathfrak{B}$, iff

(a) $A \subseteq B$, and

(b) for every n-ary $R \in S$, every n-ary $f \in S$ and every constant $c \in S$,

$$R^{\mathfrak{A}} = R^{\mathfrak{B}} \cap A^n, \qquad f^{\mathfrak{A}} = f^{\mathfrak{B}} \restriction A^n, \qquad c^{\mathfrak{A}} = c^{\mathfrak{B}}.$$

Definition 22 (S-closed subset, generated substructure $[X]^{\mathfrak{B}}$)**.** $X \subseteq B$ is *S-closed in* $\mathfrak{B}$ iff $X \neq \emptyset$, $c^{\mathfrak{B}} \in X$ for every constant $c \in S$, and $f^{\mathfrak{B}}(a_1, \dots, a_n) \in X$ whenever $f \in S$ is n-ary and $a_1, \dots, a_n \in X$.

If $\mathfrak{A} \subseteq \mathfrak{B}$ then A is S-closed in $\mathfrak{B}$. Conversely, if X is S-closed in $\mathfrak{B}$ then the clauses of Definition 21(b) determine exactly one substructure with domain X. We write it $[X]^{\mathfrak{B}}$, the *substructure generated by* X.

For example, with $S_{\mathrm{gr}} := \{\circ, e\}$ the language of groups without an inverse symbol, $\{2n \mid n \in \mathbb{N}\}$ is S_{gr}-closed in $(\mathbb{Z}, +, 0)$, while $\{2n+1 \mid n \in \mathbb{N}\}$ is not, since $3+3$ is even.

Lemma 14 (E&F III.5.5: substructures preserve quantifier-free formulas)**.** Let $\mathfrak{A} \subseteq \mathfrak{B}$ and let β be an assignment in $\mathfrak{A}$. Then

$$(\mathfrak{A}, \beta)(t) = (\mathfrak{B}, \beta)(t) \quad \text{for every } S\text{-term } t,$$

and for every quantifier-free S-formula φ,

$$(\mathfrak{A}, \beta) \models \varphi \quad \text{iff} \quad (\mathfrak{B}, \beta) \models \varphi.$$

Proof. Both parts are the proof of Lemma 13 with π the inclusion $A \hookrightarrow B$. It is injective and satisfies the three compatibility clauses by Definition 21(b), which is all the term case and the atomic cases use. The quantifier case is dropped rather than adapted, because the inclusion is surjective only when $A = B$. □

Remark 7. $(\mathbb{Z}, +, 0) \subseteq (\mathbb{Q}, +, 0)$ and $(\mathbb{Q}, +, 0) \models \forall v_0 \exists v_1\, v_1 + v_1 \equiv v_0$, while $(\mathbb{Z}, +, 0)$ does not. So quantified formulas need not transfer in either direction.

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Some quantified formulas do transfer downwards, and the following class isolates them.

Definition 23 (E&F III.5.6: universal formulas)**.** The *universal* formulas are those derivable in the calculus

$$\frac{}{\varphi}\ (\varphi \text{ quantifier-free}), \qquad \frac{\varphi,\ \psi}{(\varphi * \psi)}\ (* = \wedge, \vee), \qquad \frac{\varphi}{\forall x \varphi}.$$

Every universal formula is logically equivalent to one of the form $\forall x_1 \ldots \forall x_n \psi$ with ψ quantifier-free.

Lemma 15 (E&F III.5.7: Substructure Lemma)**.** Let $\mathfrak{A} \subseteq \mathfrak{B}$ and let $\varphi \in L_n^S$ be universal. Then for all $a_0, \ldots, a_{n-1} \in A$,

$$\mathfrak{B} \models \varphi[a_0, \ldots, a_{n-1}] \implies \mathfrak{A} \models \varphi[a_0, \ldots, a_{n-1}].$$

Proof. Induction over the calculus of Definition 23, proving for all assignments β in $\mathfrak{A}$ that $(\mathfrak{B}, \beta) \models \varphi$ implies $(\mathfrak{A}, \beta) \models \varphi$.

Quantifier-free φ is Lemma 14, and $\wedge, \vee$ follow from the induction hypothesis. For $\forall x \psi$, suppose $(\mathfrak{B}, \beta) \models \forall x \psi$. Then $(\mathfrak{B}, \beta\frac{b}{x}) \models \psi$ for every $b \in B$, hence in particular for every $a \in A$, since $A \subseteq B$. The induction hypothesis gives $(\mathfrak{A}, \beta\frac{a}{x}) \models \psi$ for every $a \in A$, which is $(\mathfrak{A}, \beta) \models \forall x \psi$.

The lemma follows by choosing β with $\beta(v_i) = a_i$ for $i < n$. □

The one-step move is that $\forall$ over B restricts to $\forall$ over the smaller set A. An existential quantifier moves the other way, which is why the calculus excludes it.

Corollary 2 (E&F III.5.8: universal sentences pass to substructures)**.** If $\mathfrak{A} \subseteq \mathfrak{B}$ and φ is a universal sentence, then $\mathfrak{B} \models \varphi$ implies $\mathfrak{A} \models \varphi$.

Remark 8 (Universal axiomatizability)**.** $(\mathbb{N}, +, 0)$ is a substructure of the group $(\mathbb{Z}, +, 0)$ and is not a group. By Corollary 2, no set of universal S_{gr}-sentences has exactly the groups as models. Enlarging the signature repairs this. Over $S_{\mathrm{grp}} := \{\circ, ^{-1}, e\}$ the axioms

$$\Phi_{\mathrm{grp}} := \{\forall x \forall y \forall z\, (x \circ y) \circ z \equiv x \circ (y \circ z),\ \forall x\, x \circ e \equiv x,\ \forall x\, x \circ x^{-1} \equiv e\}$$

are all universal, and substructures of groups are then exactly subgroups.

2.5 Some Simple Formalizations

Restricted quantifiers translate with different connectives:

$$\text{“for all } x \text{ with } \chi\text{: } \varphi\text{”} \ \rightsquigarrow\ \forall x(\chi \to \varphi), \qquad \text{“for some } x \text{ with } \chi\text{: } \varphi\text{”} \ \rightsquigarrow\ \exists x(\chi \wedge \varphi).$$

Cardinality

For $n \geq 2$ put

$$\varphi_{\geq n} := \exists v_0 \ldots \exists v_{n-1} \bigwedge_{i<j<n} \neg v_i \equiv v_j,$$

the finite conjunction being an abbreviation as usual. Then for every S and every S-structure $\mathfrak{A}$,

$$\mathfrak{A} \models \varphi_{\geq n} \quad \text{iff} \quad A \text{ has at least } n \text{ elements.}$$

So $\neg\varphi_{\geq n}$ says A has fewer than n elements, and $\varphi_{\geq n} \wedge \neg\varphi_{\geq n+1}$ says exactly n. Setting

$$\Phi_\infty := \{\varphi_{\geq n} \mid n \geq 2\},$$

the models of Φ_∞ are exactly the infinite structures. Note that no single sentence is used, only the infinite set.

Standard axiom systems

Definition 24 (Axiom system, Mod)**.** For a symbol set S, an S*-axiom system* is a set $\Phi \subseteq L_0^S$ of S-sentences. Its *model class* is $\mathrm{Mod}(\Phi) := \{\mathfrak{A} \mid \mathfrak{A} \models \Phi\}$, and Φ *axiomatizes* a class K of S-structures iff $\mathrm{Mod}(\Phi) = K$.

All of the following are first-order.

- **Groups.** $S_{\mathrm{gr}} := \{\circ, e\}$ ($\circ$ binary, e a constant):

  $$\Phi_{\mathrm{gr}} := \Big\{\ \forall x \forall y \forall z\ (x \circ y) \circ z \equiv x \circ (y \circ z), \quad \forall x\ x \circ e \equiv x, \quad \forall x \exists y\ x \circ y \equiv e\ \Big\}.$$

  $S_{\mathrm{grp}} := \{\circ, ^{-1}, e\}$, and Φ_{grp} is as in Remark 8.

- **Orderings.** $S_{\mathrm{ord}} := \{<\}$:

  $$\Phi_{\mathrm{ord}} := \Big\{\ \forall x\ \neg x < x, \quad \forall x \forall y \forall z\ ((x < y \wedge y < z) \to x < z), \quad \forall x \forall y\ (x < y \vee x \equiv y \vee y < x)\ \Big\}.$$

  $S_{\mathrm{pord}} := S_{\mathrm{ord}}$:

  $$\Phi_{\mathrm{pord}} := \Big\{\ \exists x \exists y\ x < y, \quad \forall x\ \neg x < x, \quad \forall x \forall y \forall z\ ((x < y \wedge y < z) \to x < z),\ \Big\} \cup$$

  $$\Big\{\ \forall x \forall y \big((\exists u\, (x < u \vee u < x) \wedge \exists v\, (y < v \vee v < y)) \to (x < y \vee x \equiv y \vee y < x)\big)\ \Big\}.$$

- **Fields.** $S_{\mathrm{ar}} := \{+, \cdot, 0, 1\}$ ($+, \cdot$ binary, $0, 1$ constants). Φ_{fd}:

$$\begin{array}{ll}
\forall x \forall y \forall z\ (x+y)+z \equiv x+(y+z), & \forall x \forall y\ x+y \equiv y+x, \\
\forall x\ x+0 \equiv x, & \forall x \exists y\ x+y \equiv 0, \\
\forall x \forall y \forall z\ (x \cdot y) \cdot z \equiv x \cdot (y \cdot z), & \forall x \forall y\ x \cdot y \equiv y \cdot x, \\
\forall x\ x \cdot 1 \equiv x, & \forall x\ (\neg x \equiv 0 \rightarrow \exists y\ x \cdot y \equiv 1), \\
\forall x \forall y \forall z\ x \cdot (y+z) \equiv (x \cdot y)+(x \cdot z), & \neg\, 0 \equiv 1.
\end{array}$$

  $S_{\mathrm{ar}}^{<} := S_{\mathrm{ar}} \cup \{<\}$. $\Phi_{\mathrm{ofd}} := \Phi_{\mathrm{fd}} \cup \Phi_{\mathrm{ord}} \cup$

$$\left\{\ \forall x \forall y \forall z\ (x<y \rightarrow x+z<y+z), \quad \forall x \forall y \forall z\ ((x<y \wedge 0<z) \rightarrow x \cdot z<y \cdot z)\ \right\}.$$

- **Graphs.** $S_{\mathrm{dgph}} := S_{\mathrm{gph}} := \{R\}$ (R binary):

$$\Phi_{\mathrm{dgph}} := \{\forall x\ \neg Rxx\}, \qquad \Phi_{\mathrm{gph}} := \Phi_{\mathrm{dgph}} \cup \{\forall x \forall y\ (Rxy \leftrightarrow Ryx)\}.$$

2.6 Some Remarks on Formalizability

Partial functions

Function symbols are interpreted by total functions, so division on $\mathbb{R}$ is not directly a structure. Repairs: totalize by fiat, e.g. $r/0 := 0$, carried through every statement about division; replace the function by its graph, the ternary relation $\{(a,b,c) \mid b \neq 0,\ a/b=c\}$, translating function-talk into relation-talk; or admit partial function symbols into the logic itself, which only complicates it without adding anything the first two don't already give.

Many-sorted structures

A vector space has two sorts, scalars and vectors: $\mathfrak{V} = (F, V, +^F, \cdot^F, 0^F, 1^F, \circ^V, e^V, *^{F,V})$, with $(F, +^F, \cdot^F, 0^F, 1^F)$ the field of scalars, $(V, \circ^V, e^V)$ the additive group of vectors, $*^{F,V}$ scalar multiplication. Two treatments: *many-sorted languages*, with separate variables $u_0, u_1, \ldots$ for scalars and $w_0, w_1, \ldots$ for vectors, each quantifier ranging over its own domain; or *sort reduction*, staying one-sorted over $F \cup V$, adding unary relations F, V naming the sorts, extending each operation arbitrarily to $(F \cup V)^n$, and relativizing every quantifier to its sort, e.g. associativity of scalar addition becomes $\forall x \forall y \forall z\, ((Fx \wedge Fy \wedge Fz) \rightarrow (x+y)+z \equiv x+(y+z))$; since quantifiers are relativized, the arbitrary extension is never seen.

Limits of formalizability

Torsion groups. "Every element has finite order" would need $\forall x (x \equiv e \vee x \circ x \equiv e \vee (x \circ x) \circ x \equiv e \vee \ldots)$, an infinite disjunction, which is not a formula. No set of first-order sentences has exactly the torsion groups as models.

Peano's axioms. For $\mathfrak{N}_\sigma = (\mathbb{N}, \sigma, 0)$ with σ the successor function, the axioms are that 0 is not a successor, that σ is injective, and induction. The first two are first-order:

$$\text{(P1)}\ \ \forall x \neg \sigma x \equiv 0, \qquad \text{(P2)}\ \ \forall x \forall y (\sigma x \equiv \sigma y \rightarrow x \equiv y).$$

Induction quantifies over arbitrary subsets of the domain, so its formalization

$$\text{(P3)}\ \ \forall X \big((X0 \wedge \forall x (Xx \rightarrow X\sigma x)) \rightarrow \forall y\, Xy\big)$$

is second-order.

Theorem 2 (E&F III.7.4: Dedekind's Theorem)**.** Every structure $\mathfrak{A} = (A, \sigma^{\mathfrak{A}}, 0^{\mathfrak{A}})$ satisfying (P1)–(P3) is isomorphic to $\mathfrak{N}_\sigma$.

Proof. An isomorphism $\pi : \mathfrak{N}_\sigma \cong \mathfrak{A}$ must satisfy $\pi(0) = 0^{\mathfrak{A}}$ and $\pi(n+1) = \sigma^{\mathfrak{A}}(\pi(n))$ for every $n \in \mathbb{N}$; take these as the recursive definition of a function $\pi : \mathbb{N} \to A$. This is ordinary recursion on $\mathbb{N}$ and needs nothing about $\mathfrak{A}$. The compatibility conditions of an isomorphism hold by this construction outright, so only bijectivity of π remains, and its two directions rest on two different inductions.

Surjectivity: induction inside $\mathfrak{A}$*, i.e. (P3) itself.* (P3) is a statement about *every* subset $X \subseteq A$; instantiate it at $X := \operatorname{ran}(\pi)$. By construction $0^{\mathfrak{A}} = \pi(0) \in X$, and if $\pi(n) \in X$ then $\sigma^{\mathfrak{A}}(\pi(n)) = \pi(n+1) \in X$, so X is closed under $\sigma^{\mathfrak{A}}$. Since $\mathfrak{A} \models$ (P3), this forces $X = A$, i.e. π is onto. *Injectivity: ordinary induction on* $\mathbb{N}$

$$P(n) :\iff \text{ for all } m \neq n,\ \pi(m) \neq \pi(n).$$

$P(0)$: if $m \neq 0$ then $m = k+1$ for some k, so $\pi(m) = \sigma^{\mathfrak{A}}(\pi(k)) \neq 0^{\mathfrak{A}} = \pi(0)$ by (P1) (applied in $\mathfrak{A}$, since $\mathfrak{A} \models$ (P1)). $P(n) \Rightarrow P(n+1)$: let $m \neq n+1$. If $m = 0$, then $\pi(m) = 0^{\mathfrak{A}} \neq \pi(n+1)$, by the very same instance of (P1) used for $P(0)$ (with the roles of the two sides swapped). If $m = k+1$, then $k \neq n$, so $\pi(k) \neq \pi(n)$ by $P(n)$, and (P2) contrapositively gives $\sigma^{\mathfrak{A}}(\pi(k)) \neq \sigma^{\mathfrak{A}}(\pi(n))$, i.e. $\pi(m) \neq \pi(n+1)$. □

2.7 Substitution

Replacing a free variable by a term can change the meaning of a formula if the term's variables get captured by a quantifier. In $\mathfrak{N}$ the formula $\varphi := \exists z\, z + z \equiv x$ says that x is even. Replacing x by y gives $\exists z\, z + z \equiv y$, which says y is even, as intended. Replacing x by z gives $\exists z\, z + z \equiv z$, which is valid under every assignment. The occurrence of z was captured by the quantifier. The repair is to rename the bound variable before substituting.

The definitions below substitute several variables at once. Throughout, $x_0, \ldots, x_r$ are pairwise distinct variables, $t_0, \ldots, t_r$ are terms, and $\frac{\bar{t}}{\bar{x}}$ abbreviates $\frac{t_0 \ldots t_r}{x_0 \ldots x_r}$. Note that these definitions are meta-theoretic, not part of the formal language.

Definition 25 (E&F III.8.1: substitution in terms)**.** For terms, by recursion:

$$\begin{aligned} x\frac{\bar{t}}{\bar{x}} &:= \begin{cases} t_i & \text{if } x = x_i, \\ x & \text{if } x \neq x_0, \ldots, x \neq x_r, \end{cases} \\ c\frac{\bar{t}}{\bar{x}} &:= c, \\ [f t'_1 \ldots t'_n]\frac{\bar{t}}{\bar{x}} &:= f\, t'_1\frac{\bar{t}}{\bar{x}} \ldots t'_n\frac{\bar{t}}{\bar{x}}. \end{aligned}$$

Definition 26 (E&F III.8.2: substitution in formulas)**.** For formulas, by recursion:

$$\begin{aligned} [t'_1 \equiv t'_2]\frac{\bar{t}}{\bar{x}} &:= t'_1\frac{\bar{t}}{\bar{x}} \equiv t'_2\frac{\bar{t}}{\bar{x}}, \\ [R t'_1 \ldots t'_n]\frac{\bar{t}}{\bar{x}} &:= R\, t'_1\frac{\bar{t}}{\bar{x}} \ldots t'_n\frac{\bar{t}}{\bar{x}}, \\ [\neg\varphi]\frac{\bar{t}}{\bar{x}} &:= \neg\big[\varphi\frac{\bar{t}}{\bar{x}}\big], \\ [(\varphi \vee \psi)]\frac{\bar{t}}{\bar{x}} &:= \big(\varphi\frac{\bar{t}}{\bar{x}} \vee \psi\frac{\bar{t}}{\bar{x}}\big). \end{aligned}$$

For $\exists x\varphi$, let $x_{i_1}, \ldots, x_{i_s}$ with $i_1 < \cdots < i_s$ be exactly those x_i satisfying

$$x_i \in \operatorname{free}(\exists x\varphi) \quad \text{and} \quad x_i \neq t_i,$$

so that in particular $x \neq x_{i_1}, \dots, x \neq x_{i_s}$. Then

$$[\exists x \varphi]\frac{\bar{t}}{\bar{x}} := \exists u \Big[\varphi \frac{t_{i_1} \dots t_{i_s}\ u}{x_{i_1} \dots x_{i_s}\ x}\Big],$$

where $u := x$ if x occurs in none of $t_{i_1}, \dots, t_{i_s}$, and otherwise u is the first variable in the list $v_0, v_1, \dots$ occurring in none of $\varphi, t_{i_1}, \dots, t_{i_s}$.

Three things are happening in the quantifier clause.

- The condition $x_i \in \mathrm{free}(\exists x \varphi)$ and $x_i \neq t_i$ discards the substitutions that would do nothing. In the other clauses this happens automatically.
- Renaming x to u prevents capture. The choice of u avoids every variable of φ and of the terms being substituted, so no variable of $t_{i_1}, \dots, t_{i_s}$ can fall under the new quantifier.
- Fixing u as the *first* unused variable rather than any unused one makes substitution a function. Otherwise it would be a family of choices, and $\varphi \frac{\bar{t}}{\bar{x}}$ would not denote a single formula.

Lemma 16 (E&F III.8.3: Substitution Lemma)**.** Let $\mathfrak{I} = (\mathfrak{A}, \beta)$. For every term t',

$$\mathfrak{I}\Big(t' \frac{\bar{t}}{\bar{x}}\Big) = \mathfrak{I}\frac{\mathfrak{I}(t_0) \dots \mathfrak{I}(t_r)}{x_0 \dots x_r}(t'),$$

and for every formula φ,

$$\mathfrak{I} \models \varphi \frac{\bar{t}}{\bar{x}} \quad \text{iff} \quad \mathfrak{I}\frac{\mathfrak{I}(t_0) \dots \mathfrak{I}(t_r)}{x_0 \dots x_r} \models \varphi.$$

Read the right-hand sides as follows: instead of substituting the terms into φ, evaluate the terms first and assign their values to $x_0, \dots, x_r$. The lemma says the two give the same answer, which is precisely the statement that Definition 26 substitutes correctly.

Proof. Induction on terms and formulas, following the clauses of Definitions 25 and 26. Throughout write $\mathfrak{I}^{\bar{t}} := \mathfrak{I}\frac{\mathfrak{I}(t_0) \dots \mathfrak{I}(t_r)}{x_0 \dots x_r}$ for the right-hand assignment.

Terms. For $t' = x$: if $x \neq x_0, \dots, x \neq x_r$ then $x \frac{\bar{t}}{\bar{x}} = x$, and $\mathfrak{I}(x) = \beta(x) = \mathfrak{I}^{\bar{t}}(x)$ since $\mathfrak{I}^{\bar{t}}$ agrees with $\mathfrak{I}$ off $x_0, \dots, x_r$. If $x = x_i$ then $x \frac{\bar{t}}{\bar{x}} = t_i$, and $\mathfrak{I}(t_i) = \mathfrak{I}^{\bar{t}}(x_i)$ by definition of $\mathfrak{I}^{\bar{t}}$ and Definition 13. The cases $t' = c$ and $t' = f t'_1 \dots t'_n$ are immediate from the definition and the induction hypothesis respectively, exactly as in the term case of Lemma 11(a).

Atomic formulas. Immediate from the term case: e.g. for $\varphi = R t'_1 \dots t'_n$,

$$\begin{aligned} \mathfrak{I} \models [R t'_1 \dots t'_n]\frac{\bar{t}}{\bar{x}} &\iff \mathfrak{I} \models R\, t'_1 \frac{\bar{t}}{\bar{x}} \dots t'_n \frac{\bar{t}}{\bar{x}} \\ &\iff R^{\mathfrak{A}} \mathfrak{I}\big(t'_1 \tfrac{\bar{t}}{\bar{x}}\big) \dots \mathfrak{I}\big(t'_n \tfrac{\bar{t}}{\bar{x}}\big) \\ &\iff R^{\mathfrak{A}} \mathfrak{I}^{\bar{t}}(t'_1) \dots \mathfrak{I}^{\bar{t}}(t'_n) && \text{(term case)} \\ &\iff \mathfrak{I}^{\bar{t}} \models R t'_1 \dots t'_n, \end{aligned}$$

$\neg$ and $\vee$ pass straight through the induction hypothesis.

Quantifiers, $\varphi = \exists x \psi$. Let $x_{i_1}, \dots, x_{i_s}$ and u be as in Definition 26. Then

$$\begin{aligned} \mathfrak{I} \models [\exists x \psi]\frac{\bar{t}}{\bar{x}} &\iff \mathfrak{I} \models \exists u \Big[\psi \frac{t_{i_1} \dots t_{i_s}\ u}{x_{i_1} \dots x_{i_s}\ x}\Big] \\ &\iff \text{there is } a \in A \text{ with } \mathfrak{I}\frac{a}{u} \models \psi \frac{t_{i_1} \dots t_{i_s}\ u}{x_{i_1} \dots x_{i_s}\ x} \\ &\iff \text{there is } a \in A \text{ with } \Big(\mathfrak{I}\frac{a}{u}\Big)\frac{(\mathfrak{I}\frac{a}{u})(t_{i_1}) \dots (\mathfrak{I}\frac{a}{u})(t_{i_s})\ \ (\mathfrak{I}\frac{a}{u})(u)}{x_{i_1} \dots x_{i_s}\ \ x} \models \psi && \text{(induction hypothesis)} \end{aligned}$$

Since u occurs in none of $t_{i_1}, \dots, t_{i_s}$ (by the choice of u), the Coincidence Lemma gives $(\mathfrak{I}\frac{a}{u})(t_{i_k}) = \mathfrak{I}(t_{i_k})$ for each k, while $(\mathfrak{I}\frac{a}{u})(u) = a$ outright. So the assignment above is $\left(\mathfrak{I}\frac{a}{u}\right)\frac{\mathfrak{I}(t_{i_1})\dots\mathfrak{I}(t_{i_s})\ \ a}{x_{i_1}\dots x_{i_s}\ \ x}$. Since $u = x$ or u occurs in neither ψ nor $t_{i_1}, \dots, t_{i_s}$ (again by the choice of u), a second application of the Coincidence Lemma drops the outer $\frac{a}{u}$ without affecting $\models \psi$, leaving

$$\text{there is } a \in A \text{ with } \mathfrak{I}\frac{\mathfrak{I}(t_{i_1})\dots\mathfrak{I}(t_{i_s})\ \ a}{x_{i_1}\dots x_{i_s}\ \ x} \models \psi,$$

that is, since $x \neq x_{i_1}, \dots, x \neq x_{i_s}$, $\left(\mathfrak{I}\frac{\mathfrak{I}(t_{i_1})\dots\mathfrak{I}(t_{i_s})}{x_{i_1}\dots x_{i_s}}\right)\frac{a}{x} \models \psi$ for some $a \in A$, i.e. $\mathfrak{I}\frac{\mathfrak{I}(t_{i_1})\dots\mathfrak{I}(t_{i_s})}{x_{i_1}\dots x_{i_s}} \models \exists x\psi$.

Finally, restore the dropped coordinates $i \neq i_1, \dots, i_s$: either $x_i \notin \text{free}(\exists x\psi)$, so by the Coincidence Lemma its assigned value cannot affect $\models \exists x\psi$, or $x_i = t_i$, so assigning $\mathfrak{I}(t_i) = \mathfrak{I}(x_i) = \beta(x_i)$ reproduces the original assignment at x_i unchanged. Either way,

$$\mathfrak{I}\frac{\mathfrak{I}(t_{i_1})\dots\mathfrak{I}(t_{i_s})}{x_{i_1}\dots x_{i_s}} \models \exists x\psi \quad \text{iff} \quad \mathfrak{I}^{\bar{t}} \models \exists x\psi,$$

which combined with the previous display gives $\mathfrak{I} \models [\exists x\psi]\frac{\bar{t}}{\bar{x}}$ iff $\mathfrak{I}^{\bar{t}} \models \exists x\psi$, as required. $\square$

Lemma 17 (E&F III.8.4: syntactic properties of substitution)**.** (a) For every permutation π of $0, \dots, r$,

$$\varphi\frac{t_0\dots t_r}{x_0\dots x_r} = \varphi\frac{t_{\pi(0)}\dots t_{\pi(r)}}{x_{\pi(0)}\dots x_{\pi(r)}}.$$

(b) If $x_i = t_i$ for some i, then

$$\varphi\frac{t_0\dots t_r}{x_0\dots x_r} = \varphi\frac{t_0\dots t_{i-1}t_{i+1}\dots t_r}{x_0\dots x_{i-1}x_{i+1}\dots x_r};$$

in particular $\varphi\frac{x}{x} = \varphi$.

(c) For every variable y:

(i) if $y \in \text{var}\left(t\frac{\bar{t}}{\bar{x}}\right)$, then $y \in \text{var}(t_0) \cup \dots \cup \text{var}(t_r)$, or ($y \in \text{var}(t)$ and $y \neq x_0, \dots, y \neq x_r$);

(ii) if $y \in \text{free}\left(\varphi\frac{\bar{t}}{\bar{x}}\right)$, then $y \in \text{var}(t_0) \cup \dots \cup \text{var}(t_r)$, or ($y \in \text{free}(\varphi)$ and $y \neq x_0, \dots, y \neq x_r$).

Corollary 3 (E&F III.8.5: substitution into sentences)**.** Suppose $\text{free}(\varphi) \subseteq \{x_0, \dots, x_r\}$. If $\text{var}(t_i) \subseteq \{v_0, \dots, v_{n-1}\}$ for every i, then $\varphi\frac{\bar{t}}{\bar{x}} \in L_n^S$. In particular, $\varphi\frac{c_0\dots c_r}{x_0\dots x_r}$ is a sentence for constants $c_0, \dots, c_r$.

Definition 27 (E&F III.8.6: rank)**.** The *rank* $\text{rk}(\varphi)$ of a formula, the number of connectives and quantifiers it contains, is defined by recursion on formulas:

$$\begin{aligned} \text{rk}(\varphi) &:= 0 \quad \text{if } \varphi \text{ is atomic}, \\ \text{rk}(\neg\varphi) &:= \text{rk}(\varphi) + 1, \\ \text{rk}((\varphi \vee \psi)) &:= \text{rk}(\varphi) + \text{rk}(\psi) + 1, \\ \text{rk}(\exists x\varphi) &:= \text{rk}(\varphi) + 1. \end{aligned}$$

Lemma 18 (E&F III.8.7: rank is invariant under substitution)**.** $\text{rk}\left(\varphi\frac{\bar{t}}{\bar{x}}\right) = \text{rk}(\varphi)$.

3 Hermes' Sequent Calculus

Ebbinghaus–Flum–Thomas, Ch. IV · 26/08/2026

E&F write a sequent as a string $\Gamma\varphi$ and its derivability as $\vdash \Gamma\varphi$, keeping sequents inside the string calculi of Ch. II. I write $\Gamma \vdash \varphi$ with Γ a finite *set* as that is the modern convention.

3.1 Sequents and Derivability

Definition 28 (sequent; cf. E&F IV.1)**.** A *sequent* is a pair (Γ, φ) with Γ a finite set of S-formulas, the *antecedent*, and φ an S-formula, the *succedent*. It is written $\Gamma \vdash \varphi$. We abbreviate $\Gamma \cup \{\psi\}$ by Γ, ψ; write $\psi_1, \ldots, \psi_n \vdash \varphi$ for $\{\psi_1, \ldots, \psi_n\} \vdash \varphi$; and write $\vdash \varphi$ for $\emptyset \vdash \varphi$.

Definition 29 (rule, correctness)**.** A *rule* consists of finitely many premise sequents, one conclusion sequent, and possibly a side condition; an *instance* is the result of substituting concrete formulas, terms and variables for the schematic letters so that the side condition holds. A sequent $\Gamma \vdash \varphi$ is *correct* iff $\Gamma \models \varphi$. A rule is *correct* iff in every instance, if all premises are correct then so is the conclusion; for a rule with no premises this requires every instance of the conclusion to be correct outright.

Definition 30 (the rules of $\mathcal{S}$; E&F IV.2, IV.4)**. Structural rules.**

(Ant) $\dfrac{\Gamma \vdash \varphi}{\Gamma' \vdash \varphi}$ if $\Gamma \subseteq \Gamma'$. Correct since $\Gamma \models \varphi$ and $\Gamma \subseteq \Gamma'$ give $\Gamma' \models \varphi$.

(Assm) $\dfrac{}{\Gamma \vdash \varphi}$ if $\varphi \in \Gamma$. Correct since $\Phi \models \varphi$ always holds for $\varphi \in \Phi$.

Connective rules (restricted to $\neg, \vee$; cf. Remark 6).

(PC) $\dfrac{\Gamma, \psi \vdash \varphi \qquad \Gamma, \neg\psi \vdash \varphi}{\Gamma \vdash \varphi}$ (proof by cases). Correct: every model of Γ satisfies ψ or $\neg\psi$, hence φ either way.

(Ctr) $\dfrac{\Gamma, \neg\varphi \vdash \psi \qquad \Gamma, \neg\varphi \vdash \neg\psi}{\Gamma \vdash \varphi}$ (contradiction rule). Correct: no interpretation models $\Gamma, \neg\varphi$, so every model of Γ models φ.

($\vee$A) $\dfrac{\Gamma, \varphi \vdash \chi \qquad \Gamma, \psi \vdash \chi}{\Gamma, (\varphi \vee \psi) \vdash \chi}$.

($\vee$S) $\dfrac{\Gamma \vdash \varphi}{\Gamma \vdash (\varphi \vee \psi)}$ and $\dfrac{\Gamma \vdash \varphi}{\Gamma \vdash (\psi \vee \varphi)}$.

Quantifier rules (for $\exists$ only; cf. Remark 6).

($\exists$S) $\dfrac{\Gamma \vdash \varphi\frac{t}{x}}{\Gamma \vdash \exists x\varphi}$ ($\exists$-introduction in the succedent). Correct by the Substitution Lemma 16.

($\exists$A) $\dfrac{\Gamma, \varphi\frac{y}{x} \vdash \psi}{\Gamma, \exists x\varphi \vdash \psi}$ if y is not free in $\Gamma \cup \{\exists x\varphi, \psi\}$ ($\exists$-introduction in the antecedent). Correct by the Coincidence Lemma 11 and the Substitution Lemma 16.

Equality rules.

($\equiv$) $\dfrac{}{\vdash t \equiv t}$.

(Sub) $\dfrac{\Gamma \vdash \varphi\frac{t}{x}}{\Gamma, t \equiv t' \vdash \varphi\frac{t'}{x}}$ (substitution rule for equality). Correct by two applications of the Substitution Lemma 16, using $\mathfrak{I}(t) = \mathfrak{I}(t')$.

The side condition in ($\exists$A) cannot be dropped. Taking $\varphi := x \equiv fy$ and $\psi := y \equiv fy$, the premise $\varphi\frac{y}{x} \vdash \psi$ reads $y \equiv fy \vdash y \equiv fy$ and is correct, while the conclusion $\exists x\, x \equiv fy \vdash y \equiv fy$ is not: interpret the domain as $\mathbb{N}$ with f the successor function and y as 0. Here y is free in ψ, so the condition fails, as it must.

Definition 31 (derivation, derivability; E&F IV.1.1)**.** A *derivation* in $\mathcal{S}$ is a finite list of sequents, each of which is the conclusion of an instance of a rule of $\mathcal{S}$ whose premises occur earlier in the list. A sequent is *derivable* if it occurs in some derivation. We write $\Gamma \vdash \varphi$ both for the sequent and for the assertion that it is derivable; context separates the two.

For an arbitrary, possibly infinite, set $\Phi \subseteq L^S$, a formula φ is *formally provable* from Φ, written $\Phi \vdash \varphi$, iff $\Gamma \vdash \varphi$ is derivable for some finite $\Gamma \subseteq \Phi$. Antecedents of sequents stay finite, so no derivation ever sees more than finitely much of Φ. For finite Φ the two readings agree: take $\Gamma := \Phi$ in one direction, and apply (Ant) in the other.

Requiring antecedents to be finite is what keeps ($\exists$A) always applicable: only finitely many variables occur free in a finite Γ, so a fresh y exists. Over an infinite Φ this can fail.

Definition 32 (derivable rule)**.** Write $D(\mathcal{C})$ for the set of sequents derivable in a calculus $\mathcal{C}$. A rule

$$R \;=\; \frac{\Gamma_1 \vdash \varphi_1 \quad \cdots \quad \Gamma_n \vdash \varphi_n}{\Gamma \vdash \varphi}$$

is *derivable in* $\mathcal{S}$ iff every instance of it satisfies

$$\Gamma_1 \vdash \varphi_1,\ \ldots,\ \Gamma_n \vdash \varphi_n \in D(\mathcal{S}) \quad\Longrightarrow\quad \Gamma \vdash \varphi \in D(\mathcal{S}),$$

equivalently iff $D(\mathcal{S} \cup \{R\}) = D(\mathcal{S})$.

3.2 A Few Derivable Rules

Every sequent $\vdash (\varphi \vee \neg\varphi)$ ("tertium non datur") is derivable:

1.	$\varphi \vdash \varphi$	(Assm)
2.	$\varphi \vdash (\varphi \vee \neg\varphi)$	($\vee$S) on 1
3.	$\neg\varphi \vdash \neg\varphi$	(Assm)
4.	$\neg\varphi \vdash (\varphi \vee \neg\varphi)$	($\vee$S) on 3
5.	$\vdash (\varphi \vee \neg\varphi)$	(PC) on 2, 4.

So (TND) is derivable in the sense of Definition 32: any derivation using it can be expanded by inserting lines 1–4 before each use. A few further derivable rules, needed later, are recorded without their (routine) derivations.

- **Second contradiction rule** (Ctr′) (E&F IV.3.1): from a contradiction, anything.

$$\frac{\Gamma \vdash \psi \qquad \Gamma \vdash \neg\psi}{\Gamma \vdash \varphi}.$$

- **Chain rule** (Ch) (E&F IV.3.2): the cut rule, discharging an established hypothesis.

$$\frac{\Gamma \vdash \varphi \qquad \Gamma, \varphi \vdash \psi}{\Gamma \vdash \psi}.$$

- **Disjunctive syllogism** (E&F IV.3.4).

$$\frac{\Gamma \vdash (\varphi \vee \psi) \qquad \Gamma \vdash \neg\varphi}{\Gamma \vdash \psi}.$$

- **Modus ponens** (E&F IV.3.5).

$$\frac{\Gamma \vdash (\varphi \to \psi) \qquad \Gamma \vdash \varphi}{\Gamma \vdash \psi}, \quad \text{i.e.} \quad \frac{\Gamma \vdash (\neg\varphi \vee \psi) \qquad \Gamma \vdash \varphi}{\Gamma \vdash \psi}.$$

- **Existential generalization and universal instantiation** (E&F IV.5.1(a), 5.5(a1)).

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \exists x\varphi}, \qquad \frac{\Gamma \vdash \forall x\varphi}{\Gamma \vdash \varphi\frac{t}{x}} \quad \text{i.e.} \quad \frac{\Gamma \vdash \neg\exists x\neg\varphi}{\Gamma \vdash \varphi\frac{t}{x}}.$$

- **Symmetry and transitivity of $\equiv$** (E&F IV.5.3).

$$\frac{\Gamma \vdash t_1 \equiv t_2}{\Gamma \vdash t_2 \equiv t_1}, \qquad \frac{\Gamma \vdash t_1 \equiv t_2 \qquad \Gamma \vdash t_2 \equiv t_3}{\Gamma \vdash t_1 \equiv t_3}.$$

- **Congruence for relation and function symbols** (E&F IV.5.4). For n-ary $R \in S$ and n-ary $f \in S$,

$$\frac{\Gamma \vdash Rt_1 \ldots t_n \qquad \Gamma \vdash t_1 \equiv t'_1 \qquad \cdots \qquad \Gamma \vdash t_n \equiv t'_n}{\Gamma \vdash Rt'_1 \ldots t'_n},$$

$$\frac{\Gamma \vdash t_1 \equiv t'_1 \qquad \cdots \qquad \Gamma \vdash t_n \equiv t'_n}{\Gamma \vdash ft_1 \ldots t_n \equiv ft'_1 \ldots t'_n}.$$

3.3 Correctness of $\mathcal{S}$

Lemma 19 (E&F IV.6.1)**.** For all Φ and φ: $\Phi \vdash \varphi$ iff $\Phi_0 \vdash \varphi$ for some finite $\Phi_0 \subseteq \Phi$.

Immediate from Definition 31, which is stated in exactly this form.

Theorem 3 (E&F IV.6.2: Correctness Theorem/ Soundness Theorem)**.** For all Φ and φ, if $\Phi \vdash \varphi$ then $\Phi \models \varphi$.

Proof. Everything follows from

$$(*) \qquad \text{every } \sigma \in D(\mathcal{S}) \text{ is correct.}$$

Proof of $(*)$. Let $\sigma \in D(\mathcal{S})$ and fix a derivation $\sigma_1, \ldots, \sigma_m$ in which σ occurs. We show by strong induction on k, that is on the length of the initial segment $\sigma_1, \ldots, \sigma_k$, that

$$P(k) :\Longleftrightarrow \sigma_k \text{ is correct}$$

holds for every $1 \leq k \leq m$. This is Principle 1 with sequents in place of strings; its proof uses only that derivations are finite lists.

Let $1 \leq k \leq m$ and assume $P(j)$ for all $j < k$. By Definition 31, σ_k is the conclusion of an instance of some rule

$$R = \frac{\Gamma_1 \vdash \varphi_1 \quad \cdots \quad \Gamma_n \vdash \varphi_n}{\sigma_k} \in \mathcal{S}$$

whose premises all occur among $\sigma_1, \ldots, \sigma_{k-1}$. Each premise is therefore some σ_j with $j < k$, hence correct by the induction hypothesis. Every $R \in \mathcal{S}$ is correct, as checked rule by rule in Definition 30, so by Definition 29 its conclusion σ_k is correct, which is $P(k)$. For $n = 0$ the induction hypothesis is not used: correctness of R gives $P(k)$ outright. Since $\sigma = \sigma_k$ for some k, this proves $(*)$.

The theorem. Suppose $\Phi \vdash \varphi$. By Definition 31 there is a finite $\Gamma \subseteq \Phi$ with $\Gamma \vdash \varphi \in D(\mathcal{S})$, so $\Gamma \models \varphi$ by $(*)$. Every model of Φ is a model of Γ, since $\Gamma \subseteq \Phi$, hence a model of φ. That is, $\Phi \models \varphi$. □

3.4 Consistency

Definition 33 (E&F IV.7.1: consistent, inconsistent)**.** Φ is *consistent*, written $\operatorname{Con}\Phi$, iff there is no formula φ with $\Phi \vdash \varphi$ and $\Phi \vdash \neg\varphi$. Otherwise Φ is *inconsistent*, written $\operatorname{Inc}\Phi$; that is, $\operatorname{Inc}\Phi$ iff $\Phi \vdash \varphi$ and $\Phi \vdash \neg\varphi$ for some φ.

Lemma 20 (monotonicity)**.** If $\Phi \subseteq \Phi'$ and $\Phi \vdash \varphi$, then $\Phi' \vdash \varphi$. Hence $\operatorname{Inc}\Phi$ and $\Phi \subseteq \Phi'$ give $\operatorname{Inc}\Phi'$, and $\operatorname{Con}\Phi'$ and $\Phi \subseteq \Phi'$ give $\operatorname{Con}\Phi$.

Proof. A finite $\Gamma \subseteq \Phi$ with $\Gamma \vdash \varphi \in D(\mathcal{S})$ is also a finite subset of Φ', so Definition 31 applies to Φ' unchanged. □

Lemma 21 (E&F IV.7.2)**.** $\operatorname{Inc}\Phi$ iff $\Phi \vdash \varphi$ for every formula φ.

Proof. ($\Leftarrow$) Apply the hypothesis to any φ and to $\neg\varphi$.

($\Rightarrow$) Let $\operatorname{Inc}\Phi$, so $\Phi \vdash \psi$ and $\Phi \vdash \neg\psi$ for some ψ, and let φ be arbitrary. By Definition 31 there are finite $\Gamma_1, \Gamma_2 \subseteq \Phi$ with $\Gamma_1 \vdash \psi$ and $\Gamma_2 \vdash \neg\psi$ in $D(\mathcal{S})$. These need not coincide, so put $\Gamma := \Gamma_1 \cup \Gamma_2$, still a finite subset of Φ. By (Ant) both $\Gamma \vdash \psi$ and $\Gamma \vdash \neg\psi$ lie in $D(\mathcal{S})$, and (Ctr′) gives $\Gamma \vdash \varphi \in D(\mathcal{S})$. Hence $\Phi \vdash \varphi$. □

Corollary 4 (E&F IV.7.3)**.** $\operatorname{Con}\Phi$ iff some formula is not derivable from Φ.

Proof. Lemma 21 contraposed. To prove $\operatorname{Con}\Phi$ it therefore suffices to exhibit one underivable formula. □

Lemma 22 (E&F IV.7.4: consistency is finitary)**.** $\operatorname{Con}\Phi$ iff $\operatorname{Con}\Phi_0$ for every finite $\Phi_0 \subseteq \Phi$.

Proof. ($\Rightarrow$) By Lemma 20: a finite $\Phi_0 \subseteq \Phi$ with $\operatorname{Inc}\Phi_0$ would give $\operatorname{Inc}\Phi$.

($\Leftarrow$) Contrapositive. Assume $\operatorname{Inc}\Phi$, so $\Phi \vdash \varphi$ and $\Phi \vdash \neg\varphi$ for some φ. Applying Definition 31 to each separately gives finite sets

$$\Gamma_1 \subseteq \Phi \ \text{ with } \ \Gamma_1 \vdash \varphi \in D(\mathcal{S}), \qquad \Gamma_2 \subseteq \Phi \ \text{ with } \ \Gamma_2 \vdash \neg\varphi \in D(\mathcal{S}).$$

Put $\Phi_0 := \Gamma_1 \cup \Gamma_2$, a finite subset of Φ. Since $\Gamma_1, \Gamma_2 \subseteq \Phi_0$, Lemma 20 gives $\Phi_0 \vdash \varphi$ and $\Phi_0 \vdash \neg\varphi$, i.e. $\operatorname{Inc}\Phi_0$. □

Taking the union is the whole content of the lemma. The two derivations are unrelated, so each sees only finitely much of Φ but possibly a different part of it; neither Γ_1 nor Γ_2 need be inconsistent alone, and only their union is guaranteed to be.

Lemma 23 (E&F IV.7.5)**.** Every satisfiable set of formulas is consistent.

Proof. Contrapositive. Let $\operatorname{Inc}\Phi$, say $\Phi \vdash \varphi$ and $\Phi \vdash \neg\varphi$. By the Correctness Theorem 3, $\Phi \models \varphi$ and $\Phi \models \neg\varphi$. A model of Φ would satisfy both φ and $\neg\varphi$, so Φ has no model. □

Lemma 24 (E&F IV.7.6)**.** For all Φ and φ:

(a) $\Phi \vdash \varphi$ iff $\mathrm{Inc}\,\Phi \cup \{\neg\varphi\}$.

(b) $\Phi \vdash \neg\varphi$ iff $\mathrm{Inc}\,\Phi \cup \{\varphi\}$.

(c) If $\mathrm{Con}\,\Phi$, then $\mathrm{Con}\,\Phi \cup \{\varphi\}$ or $\mathrm{Con}\,\Phi \cup \{\neg\varphi\}$.

Proof. (a) ($\Rightarrow$) If $\Phi \vdash \varphi$ then $\Phi \cup \{\neg\varphi\} \vdash \varphi$ by Lemma 20, and $\Phi \cup \{\neg\varphi\} \vdash \neg\varphi$ by (Assm).

($\Leftarrow$) Let $\mathrm{Inc}\,\Phi \cup \{\neg\varphi\}$. By Lemma 21, $\Phi \cup \{\neg\varphi\} \vdash \varphi$, so there is a finite $\Gamma \subseteq \Phi$ with $\Gamma, \neg\varphi \vdash \varphi \in D(\mathcal{S})$, adding $\neg\varphi$ by (Ant) if it was not used. Also $\Gamma, \varphi \vdash \varphi \in D(\mathcal{S})$ by (Assm). Then (PC) with $\psi := \varphi$,

$$\frac{\Gamma, \varphi \vdash \varphi \qquad \Gamma, \neg\varphi \vdash \varphi}{\Gamma \vdash \varphi},$$

gives $\Gamma \vdash \varphi \in D(\mathcal{S})$, hence $\Phi \vdash \varphi$.

(b) The same argument with $\neg\varphi$ as succedent throughout: ($\Rightarrow$) is (Assm) with Lemma 20, and for ($\Leftarrow$), $\Gamma, \varphi \vdash \neg\varphi$ together with $\Gamma, \neg\varphi \vdash \neg\varphi$ from (Assm) gives $\Gamma \vdash \neg\varphi$ by (PC).

(c) Contrapositive. If $\mathrm{Inc}\,\Phi \cup \{\varphi\}$ and $\mathrm{Inc}\,\Phi \cup \{\neg\varphi\}$, then (b) gives $\Phi \vdash \neg\varphi$ and (a) gives $\Phi \vdash \varphi$, so $\mathrm{Inc}\,\Phi$. □

Varying the symbol set

Everything above is relative to the fixed S: formulas mean S-formulas and $\mathcal{S}$ means the calculus $\mathcal{S}_S$ over them. When several symbol sets are in play, write $\Phi \vdash_S \varphi$ and $\mathrm{Con}_S\,\Phi$ to name it explicitly.

For $S \subseteq S'$, $\Phi \subseteq L^S$ and $\varphi \in L^S$, it is a priori conceivable that $\Phi \vdash_{S'} \varphi$ while not $\Phi \vdash_S \varphi$, since a derivation over S' may pass through formulas of $L^{S'} \setminus L^S$ and eliminate them later by (Ctr), (PC) or ($\exists$S). This cannot happen (shown later), so the subscript may eventually be dropped.

Lemma 25 (E&F IV.7.7: unions of chains)**.** Let $S_0 \subseteq S_1 \subseteq S_2 \subseteq \cdots$ be symbol sets and $\Phi_0 \subseteq \Phi_1 \subseteq \Phi_2 \subseteq \cdots$ sets of formulas with $\Phi_n \subseteq L^{S_n}$ and $\mathrm{Con}_{S_n}\,\Phi_n$ for every n. Put $S := \bigcup_n S_n$ and $\Phi := \bigcup_n \Phi_n$. Then $\mathrm{Con}_S\,\Phi$.

Proof. Suppose $\mathrm{Inc}_S\,\Phi$. By Lemma 22 there is a finite $\Psi \subseteq \Phi$ with $\mathrm{Inc}_S\,\Psi$. As Ψ is finite and the Φ_n increase, $\Psi \subseteq \Phi_k$ for some k, so $\mathrm{Inc}_S\,\Phi_k$ by Lemma 20; in particular

$$\Phi_k \vdash_S v_0 \equiv v_0 \qquad \text{and} \qquad \Phi_k \vdash_S \neg\, v_0 \equiv v_0.$$

Fix S-derivations of these two. Each is a finite list of sequents and so contains only finitely many symbols, hence every formula occurring in either lies in L^{S_m} for some m, which we may take $\geq k$. Both are then derivations in $\mathcal{S}_{S_m}$, giving $\mathrm{Inc}_{S_m}\,\Phi_k$. Since $\Phi_k \subseteq \Phi_m$, Lemma 20 yields $\mathrm{Inc}_{S_m}\,\Phi_m$, contradicting $\mathrm{Con}_{S_m}\,\Phi_m$. □

Finiteness of derivations is what makes this work: an inconsistency in the limit must already be visible at some finite stage Φ_m over some finite stage S_m.

4 The Completeness Theorem

The goal is the converse of Theorem 3:

$$(*) \qquad \text{for all } \Phi, \varphi : \quad \Phi \models \varphi \implies \Phi \vdash \varphi.$$

It follows from

$$(**)\qquad \text{every consistent set of formulas is satisfiable.}$$

Assume $(**)$ and suppose $\Phi \models \varphi$ but not $\Phi \vdash \varphi$. By Lemma 24(a), not $\Phi \vdash \varphi$ gives $\mathrm{Con}\, \Phi \cup \{\neg\varphi\}$, so $\Phi \cup \{\neg\varphi\}$ is satisfiable by $(**)$. But $\Phi \models \varphi$ gives that $\Phi \cup \{\neg\varphi\}$ is unsatisfiable, by Lemma 10. Contradiction.

So $(**)$ is sufficient for $(*)$.

4.1 Henkin's Theorem

A consistent Φ supplies only syntactic information, so the model must be built from syntax. The first attempt takes the domain to be T^S itself, with $\beta(v_i) := v_i$, $f^{\mathfrak{A}}(t) := ft$, and $R^{\mathfrak{A}} := \{t \mid \Phi \vdash Rt\}$. Equality defeats it: for distinct variables x, y the terms fx and fy are distinct, so $\mathfrak{I}(fx) \neq \mathfrak{I}(fy)$, yet for $\Phi := \{fx \equiv fy\}$ we have $\Phi \vdash fx \equiv fy$, hence $\Phi \models fx \equiv fy$ by Theorem 3, so any model of Φ must equate the two. The repair is to quotient the terms.

Definition 34 (E&F V.1.1: the relation $\sim$)**.** For $t_1, t_2 \in T^S$, put

$$t_1 \sim t_2 \;:\Longleftrightarrow\; \Phi \vdash t_1 \equiv t_2.$$

Lemma 26 (E&F V.1.2: $\sim$ is a congruence)**.** (a) $\sim$ is an equivalence relation.

(b) $\sim$ is compatible with the symbols of S: if $t_1 \sim t_1', \ldots, t_n \sim t_n'$, then for n-ary $f \in S$

$$ft_1 \ldots t_n \sim ft_1' \ldots t_n',$$

and for n-ary $R \in S$

$$\Phi \vdash Rt_1 \ldots t_n \quad \text{iff} \quad \Phi \vdash Rt_1' \ldots t_n'.$$

Proof. Reflexivity is the rule $(\equiv)$, symmetry and transitivity are the derivable rules for $\equiv$, and (b) is congruence for function and relation symbols. For instance, symmetry: from $\Phi \vdash t_1 \equiv t_2$ the symmetry rule gives $\Phi \vdash t_2 \equiv t_1$. □

Write $\bar{t} := \{t' \in T^S \mid t \sim t'\}$ and $T^\Phi := \{\bar{t} \mid t \in T^S\}$, which is nonempty.

Definition 35 (E&F V.1.3–1.6: term structure and term interpretation)**.** The *term structure* $\mathfrak{T}^\Phi$ is the S-structure with domain T^Φ given by

$$\begin{aligned} R^{\mathfrak{T}^\Phi}\bar{t}_1 \ldots \bar{t}_n &:\Longleftrightarrow \Phi \vdash Rt_1 \ldots t_n && (R \in S\ n\text{-ary}),\\ f^{\mathfrak{T}^\Phi}(\bar{t}_1, \ldots, \bar{t}_n) &:= \overline{ft_1 \ldots t_n} && (f \in S\ n\text{-ary}),\\ c^{\mathfrak{T}^\Phi} &:= \bar{c} && (c \in S). \end{aligned}$$

These are independent of the representatives $t_1, \ldots, t_n$ by Lemma 26(b), hence well defined. With the assignment $\beta^\Phi(x) := \bar{x}$, the *term interpretation* associated with Φ is $\mathfrak{I}^\Phi := (\mathfrak{T}^\Phi, \beta^\Phi)$.

Lemma 27 (E&F V.1.7)**.** (a) $\mathfrak{I}^\Phi(t) = \bar{t}$ for every $t \in T^S$.

(b) $\mathfrak{I}^\Phi \models \varphi$ iff $\Phi \vdash \varphi$, for every *atomic* φ.

(c) $\mathfrak{I}^\Phi \models \exists x\varphi$ iff $\mathfrak{I}^\Phi \models \varphi\frac{t}{x}$ for some $t \in T^S$.

Proof. (a) Induction on terms. For $t = x$ this is $\beta^\Phi(x) = \bar{x}$ and for $t = c$ it is $c^{\mathfrak{T}^\Phi} = \bar{c}$. For $t = ft_1 \ldots t_n$,

$$\mathfrak{I}^\Phi(ft_1 \ldots t_n) = f^{\mathfrak{T}^\Phi}\big(\mathfrak{I}^\Phi(t_1), \ldots, \mathfrak{I}^\Phi(t_n)\big) = f^{\mathfrak{T}^\Phi}(\bar{t}_1, \ldots, \bar{t}_n) = \overline{ft_1 \ldots t_n},$$

using the induction hypothesis and Definition 35.

(b) Using (a) and Definition 34,

$$\mathfrak{I}^\Phi \models t_1 \equiv t_2 \iff \mathfrak{I}^\Phi(t_1) = \mathfrak{I}^\Phi(t_2) \iff \bar{t}_1 = \bar{t}_2 \iff t_1 \sim t_2 \iff \Phi \vdash t_1 \equiv t_2,$$

and $\mathfrak{I}^\Phi \models Rt_1 \dots t_n$ iff $R^{\mathfrak{T}^\Phi} \bar{t}_1 \dots \bar{t}_n$ iff $\Phi \vdash Rt_1 \dots t_n$ by Definition 35.

(c) Every element of T^Φ is $\bar{t} = \mathfrak{I}^\Phi(t)$ for some term t, by (a). So

$$\begin{aligned} \mathfrak{I}^\Phi \models \exists x \varphi &\iff \mathfrak{I}^\Phi \tfrac{a}{x} \models \varphi \ \text{ for some } a \in T^\Phi \\ &\iff \mathfrak{I}^\Phi \tfrac{\mathfrak{I}^\Phi(t)}{x} \models \varphi \ \text{ for some } t \in T^S \\ &\iff \mathfrak{I}^\Phi \models \varphi \tfrac{t}{x} \ \text{ for some } t \in T^S, \end{aligned}$$

the last step by the Substitution Lemma 16. The versions for several variables at once, and for $\forall$, follow the same way. □

So $\mathfrak{I}^\Phi$ models the atomic formulas of Φ, but not in general all of Φ. Take $S = \{R\}$ with R unary and $\Phi = \{\exists x Rx\}$. As S has no constants and no function symbols, rules (T2) and (T3) never apply, so T^S consists of the variables alone. Now if $\mathfrak{I}^\Phi \models \Phi$, then part (c) gives a term t with $\mathfrak{I}^\Phi \models Rt$, and Rt is atomic, so $\Phi \vdash Rt$ by part (b). That term is a variable y, so $\exists x Rx \vdash Ry$. But $\exists x Rx \not\models Ry$: interpret R over $\{0, 1\}$ by $R^{\mathfrak{A}} := \{0\}$ and put $\beta(y) := 1$. By Theorem 3 this refutes $\exists x Rx \vdash Ry$.

The gap is that $\mathfrak{I}^\Phi$ can only witness an existential by a *term*, while Φ here proves something exists without proving it of any particular term. Two closure conditions repair this.

Definition 36 (E&F V.1.8: negation complete/ syntactically complete, contains witnesses)**.** (a) Φ is *negation complete* iff for every formula φ, $\Phi \vdash \varphi$ or $\Phi \vdash \neg\varphi$.

(b) Φ *contains witnesses* iff for every formula $\exists x \varphi$ there is a term t with $\Phi \vdash (\exists x \varphi \to \varphi \frac{t}{x})$.

Lemma 28 (E&F V.1.9)**.** Let Φ be consistent, negation complete, and contain witnesses. Then for all φ, ψ:

(a) $\Phi \vdash \neg\varphi$ iff not $\Phi \vdash \varphi$.

(b) $\Phi \vdash (\varphi \vee \psi)$ iff $\Phi \vdash \varphi$ or $\Phi \vdash \psi$.

(c) $\Phi \vdash \exists x \varphi$ iff $\Phi \vdash \varphi \frac{t}{x}$ for some term t.

Proof. (a) If $\Phi \vdash \neg\varphi$ then not $\Phi \vdash \varphi$, else Inc Φ. If not $\Phi \vdash \varphi$ then $\Phi \vdash \neg\varphi$ by negation completeness.

(b) ($\Rightarrow$) Suppose $\Phi \vdash (\varphi \vee \psi)$ and not $\Phi \vdash \varphi$. Then $\Phi \vdash \neg\varphi$ by negation completeness, and disjunctive syllogism gives $\Phi \vdash \psi$. ($\Leftarrow$) is ($\vee$S).

(c) ($\Rightarrow$) Since Φ contains witnesses there is a term t with $\Phi \vdash (\exists x \varphi \to \varphi \frac{t}{x})$; modus ponens gives $\Phi \vdash \varphi \frac{t}{x}$. ($\Leftarrow$) is ($\exists$S). □

Each clause matches a clause of the satisfaction relation (Definition 15) with $\vdash$ in place of $\models$, which is what the induction below consumes.

Theorem 4 (E&F V.1.10: Henkin's Theorem)**.** Let Φ be consistent, negation complete, and contain witnesses. Then for every formula φ,

$$\mathfrak{I}^\Phi \models \varphi \quad \text{iff} \quad \Phi \vdash \varphi.$$

Proof. Induction on $\mathrm{rk}(\varphi)$ (Definition 27). If $\mathrm{rk}(\varphi) = 0$ then φ is atomic and this is Lemma 27(b). In reduced syntax (Remark 6) the step splits into three cases.

(1) $\varphi = \neg\psi$.

$$\mathfrak{I}^\Phi \models \neg\psi \iff \text{not } \mathfrak{I}^\Phi \models \psi \iff \text{not } \Phi \vdash \psi \iff \Phi \vdash \neg\psi,$$

by the induction hypothesis and Lemma 28(a).

(2) $\varphi = (\psi \vee \chi)$.

$$\mathfrak{I}^\Phi \models (\psi \vee \chi) \iff \mathfrak{I}^\Phi \models \psi \text{ or } \mathfrak{I}^\Phi \models \chi \iff \Phi \vdash \psi \text{ or } \Phi \vdash \chi \iff \Phi \vdash (\psi \vee \chi),$$

by the induction hypothesis and Lemma 28(b).

(3) $\varphi = \exists x\psi$.

$$\begin{aligned} \mathfrak{I}^\Phi \models \exists x\psi &\iff \mathfrak{I}^\Phi \models \psi\tfrac{t}{x} \ \text{ for some } t && \text{(Lemma 27(c))} \\ &\iff \Phi \vdash \psi\tfrac{t}{x} \ \text{ for some } t && \text{(induction hypothesis)} \\ &\iff \Phi \vdash \exists x\psi && \text{(Lemma 28(c)).} \end{aligned}$$

The induction hypothesis applies at the second step because $\mathrm{rk}(\psi\frac{t}{x}) = \mathrm{rk}(\psi) < \mathrm{rk}(\exists x\psi)$ by Lemma 18. This is why the induction is on rank rather than on the structure of φ: $\psi\frac{t}{x}$ is in general not a subformula of $\exists x\psi$. $\square$

Corollary 5 (E&F V.1.11)**.** If Φ is consistent, negation complete and contains witnesses, then $\mathfrak{I}^\Phi \models \Phi$; in particular Φ is satisfiable.

Proof. Let $\varphi \in \Phi$. Then $\{\varphi\}$ is a finite subset of Φ and $\{\varphi\} \vdash \varphi \in D(\mathcal{S})$ by (Assm), so $\Phi \vdash \varphi$ by Definition 31. Hence $\mathfrak{I}^\Phi \models \varphi$ by Theorem 4. As $\varphi \in \Phi$ was arbitrary, $\mathfrak{I}^\Phi \models \Phi$. $\square$

4.2 Satisfiability: the Countable Case

It remains to extend an arbitrary consistent Φ to one satisfying both closure conditions without destroying consistency. Here S is at most countable.

Lemma 29 (E&F V.2.1: adding witnesses)**.** Let $\Phi \subseteq L^S$ be consistent with $\mathrm{free}(\Phi) := \bigcup_{\varphi\in\Phi} \mathrm{free}(\varphi)$ finite. Then there is a consistent Ψ with $\Phi \subseteq \Psi \subseteq L^S$ which contains witnesses.

Proof. L^S is countable by Lemma 5, so list the formulas of L^S beginning with an existential quantifier as $\exists x_0\varphi_0, \exists x_1\varphi_1, \ldots$. Define ψ_n recursively: given ψ_m for $m < n$, only finitely many variables occur free in $\Phi \cup \{\psi_m \mid m < n\} \cup \{\exists x_n\varphi_n\}$, since $\mathrm{free}(\Phi)$ is finite and the rest is a finite set of formulas. Let y_n be the variable of least index not among them, and put

$$\psi_n := (\exists x_n\varphi_n \to \varphi_n\tfrac{y_n}{x_n}), \qquad \Psi := \Phi \cup \{\psi_n \mid n \in \mathbb{N}\}.$$

Ψ contains witnesses by construction. For consistency put $\Phi_n := \Phi \cup \{\psi_m \mid m < n\}$, so $\Phi_0 \subseteq \Phi_1 \subseteq \cdots$ and $\Psi = \bigcup_n \Phi_n$; by Lemma 25 (with $S_n := S$ throughout) it suffices that each Φ_n be consistent, which we show by induction on n.

$\Phi_0 = \Phi$ is consistent. Suppose Φ_n is consistent but $\Phi_{n+1} = \Phi_n \cup \{\psi_n\}$ is not. By Lemma 21 every φ is derivable from Φ_{n+1}, so for each φ there is a finite $\Gamma \subseteq \Phi_n$ with

$$\Gamma, \left(\neg\exists x_n\varphi_n \vee \varphi_n\tfrac{y_n}{x_n}\right) \vdash \varphi \ \in D(\mathcal{S}).$$

From (Assm) and ($\vee$S), $\Gamma, \neg\exists x_n\varphi_n$ derives the disjunction, so the chain rule gives $\Gamma, \neg\exists x_n\varphi_n \vdash \varphi$; the same argument on the other disjunct gives $\Gamma, \varphi_n\frac{y_n}{x_n} \vdash \varphi$. Taking φ to be a sentence, y_n is not free in $\Gamma \cup \{\exists x_n\varphi_n, \varphi\}$, so ($\exists$A) turns the latter into $\Gamma, \exists x_n\varphi_n \vdash \varphi$, and (PC) on the two yields $\Gamma \vdash \varphi$. Taking $\varphi := \exists v_0\, v_0 \equiv v_0$ and then $\varphi := \neg\exists v_0\, v_0 \equiv v_0$ gives $\mathrm{Inc}\,\Phi_n$, a contradiction. $\square$

The choice of y_n is where free(Φ) finite is used: it guarantees a fresh variable at every stage, which is what ($\exists$A) needs.

Lemma 30 (E&F V.2.2: negation completion)**.** Let $\Psi \subseteq L^S$ be consistent. Then there is a consistent, negation complete Θ with $\Psi \subseteq \Theta \subseteq L^S$.

Proof. Enumerate L^S as $\varphi_0, \varphi_1, \ldots$ (Lemma 5) and set

$$\Theta_0 := \Psi, \qquad \Theta_{n+1} := \begin{cases} \Theta_n \cup \{\varphi_n\} & \text{if } \operatorname{Con} \Theta_n \cup \{\varphi_n\}, \\ \Theta_n & \text{otherwise,} \end{cases} \qquad \Theta := \bigcup_n \Theta_n.$$

Each Θ_n is consistent by construction, so $\operatorname{Con}\Theta$ by Lemma 25. For negation completeness let $\varphi \in L^S$, say $\varphi = \varphi_n$, and suppose not $\Theta \vdash \neg\varphi$. Then $\operatorname{Con}\Theta \cup \{\varphi\}$ by Lemma 24(b), hence $\operatorname{Con}\Theta_n \cup \{\varphi\}$ by Lemma 20. So $\Theta_{n+1} = \Theta_n \cup \{\varphi\}$, giving $\varphi \in \Theta$ and $\Theta \vdash \varphi$. ☐

Corollary 6 (E&F V.2.3)**.** Every consistent Φ with free(Φ) finite is satisfiable.

Proof. Extend Φ to Ψ containing witnesses (Lemma 29), then Ψ to a consistent negation complete Θ (Lemma 30). Θ still contains witnesses, since Ψ does and $\Psi \subseteq \Theta$. By Corollary 5, Θ is satisfiable, and $\Phi \subseteq \Theta$. ☐

Theorem 5 (E&F V.2.4)**.** If S is at most countable and $\Phi \subseteq L^S$ is consistent, then Φ is satisfiable.

Proof. The restriction to remove is that free(Φ) be finite; the device is to replace free variables by new constants. Let $c_0, c_1, \ldots$ be distinct constants not in S and put $S' := S \cup \{c_n \mid n \in \mathbb{N}\}$. For $\varphi \in L^S$ let $n(\varphi)$ be least with $\operatorname{free}(\varphi) \subseteq \{v_0, \ldots, v_{n(\varphi)-1}\}$ and set

$$\varphi' := \varphi\frac{c_0 \ldots c_{n(\varphi)-1}}{v_0 \ldots v_{n(\varphi)-1}}, \qquad \Phi' := \{\varphi' \mid \varphi \in \Phi\}.$$

By Corollary 3, Φ' is a set of S'-sentences, so $\operatorname{free}(\Phi') = \emptyset$.

$\operatorname{Con}_{S'} \Phi'$. By Lemma 22 it suffices that every finite $\Phi'_0 \subseteq \Phi'$ be consistent, and by Lemma 23 it suffices that it be satisfiable. Write $\Phi'_0 = \{\varphi'_1, \ldots, \varphi'_n\}$ with $\varphi_i \in \Phi$. Then $\{\varphi_1, \ldots, \varphi_n\} \subseteq \Phi$ is consistent with finitely many free variables, so by Corollary 6 there is an S-interpretation $\mathfrak{I} = (\mathfrak{A}, \beta)$ with $\mathfrak{I} \models \{\varphi_1, \ldots, \varphi_n\}$. Expand $\mathfrak{A}$ to an S'-structure $\mathfrak{A}'$ by $c_i^{\mathfrak{A}'} := \mathfrak{I}(v_i)$. For $\mathfrak{I}' := (\mathfrak{A}', \beta)$ the Substitution Lemma 16 gives $\mathfrak{I} \models \varphi$ iff $\mathfrak{I}' \models \varphi'$ for every $\varphi \in L^S$, so $\mathfrak{I}' \models \Phi'_0$.

Conclusion. Φ' is a consistent set of S'-sentences, hence satisfiable by Corollary 6, say by $\mathfrak{I}' = (\mathfrak{A}', \beta')$. As Φ' consists of sentences we may, by the Coincidence Lemma 11, choose β' with $\beta'(v_n) = c_n^{\mathfrak{A}'}$, i.e. $\mathfrak{I}'(v_n) = \mathfrak{I}'(c_n)$. Then the Substitution Lemma gives $\mathfrak{I}' \models \varphi$ for every $\varphi \in \Phi$, since $\mathfrak{I}' \models \varphi'$. So Φ is satisfiable. ☐

4.3 Satisfiability: the General Case

Countability of S was used twice: to enumerate the $\exists$-formulas in Lemma 29, and all of L^S in Lemma 30. For witnesses the problem is that an uncountable Φ exhausts the variables, so fresh *constants* take over their role; for negation completeness the enumeration is replaced by Zorn's Lemma.

Definition 37 (E&F V.3: S^* and $W(S)$)**.** Associate with every $\varphi \in L^S$ a constant $c_\varphi \notin S$, distinct for distinct φ, and put

$$S^* := S \cup \{c_{\exists x\varphi} \mid \exists x\varphi \in L^S\}, \qquad W(S) := \left\{ \left(\exists x\varphi \to \varphi\frac{c_{\exists x\varphi}}{x}\right) \,\middle|\, \exists x\varphi \in L^S \right\}.$$

Lemma 31 (E&F V.3.4)**.** For $\Phi \subseteq L^S$: if $\mathrm{Con}_S\,\Phi$ then $\mathrm{Con}_{S^*}\,\Phi \cup W(S)$.

Proof. By Lemmas 22 and 23 it suffices to satisfy every finite $\Phi_0^* \subseteq \Phi \cup W(S)$, say

$$\Phi_0^* = \Phi_0 \cup \big\{(\exists x_1\varphi_1 \to \varphi_1\tfrac{c_1}{x_1}), \ldots, (\exists x_n\varphi_n \to \varphi_n\tfrac{c_n}{x_n})\big\}, \qquad \Phi_0 := \Phi_0^* \cap \Phi,$$

with c_i short for $c_{\exists x_i\varphi_i}$. Choose a finite $S_0 \subseteq S$ with $\Phi_0 \cup \{\exists x_i\varphi_i \mid i \leq n\} \subseteq L^{S_0}$. Then $\mathrm{Con}_{S_0}\,\Phi_0$, and $\mathrm{free}(\Phi_0)$ is finite, so Corollary 6 gives an S-interpretation $\mathfrak{I} = (\mathfrak{A}, \beta)$ with $\mathfrak{I} \models \Phi_0$. Fix $a \in A$ and choose, for $1 \leq i \leq n$, an element $a_i \in A$ with $\mathfrak{I}\frac{a_i}{x_i} \models \varphi_i$ if $\mathfrak{I} \models \exists x_i\varphi_i$, and $a_i := a$ otherwise. Expand $\mathfrak{A}$ to an S^*-structure $\mathfrak{A}^*$ by $c_i^{\mathfrak{A}^*} := a_i$, interpreting all remaining constants $c_{\exists x\varphi}$ by a, and put $\mathfrak{I}^* := (\mathfrak{A}^*, \beta)$. No $c_{\exists x\varphi}$ occurs in Φ_0, so $\mathfrak{I}^* \models \Phi_0$ by the Coincidence Lemma 11. And if $\mathfrak{I}^* \models \exists x_i\varphi_i$ then $\mathfrak{I}^*\frac{a_i}{x_i} \models \varphi_i$ by the choice of a_i, hence $\mathfrak{I}^* \models \varphi_i\frac{c_i}{x_i}$ by the Substitution Lemma 16, since $a_i = \mathfrak{I}^*(c_i)$. So $\mathfrak{I}^* \models \Phi_0^*$. $\square$

Lemma 32 (E&F V.3.1: witnesses, general case)**.** Let $\Phi \subseteq L^S$ with $\mathrm{Con}_S\,\Phi$. Then there are $S' \supseteq S$ and Ψ with $\Phi \subseteq \Psi \subseteq L^{S'}$, $\mathrm{Con}_{S'}\,\Psi$, and Ψ containing witnesses with respect to S'.

Proof. Define $S_0 := S$, $S_{n+1} := (S_n)^*$, and $\Phi_0 := \Phi$, $\Phi_{n+1} := \Phi_n \cup W(S_n)$, so that $S_n \subseteq S_{n+1}$, $\Phi_n \subseteq \Phi_{n+1}$ and $\Phi_n \subseteq L^{S_n}$. Put $S' := \bigcup_n S_n$ and $\Psi := \bigcup_n \Phi_n$. Lemma 31 gives $\mathrm{Con}_{S_n}\,\Phi_n$ by induction on n, so $\mathrm{Con}_{S'}\,\Psi$ by Lemma 25. For witnesses let $\exists x\varphi \in L^{S'}$; being a finite string it lies in L^{S_n} for some n, so $(\exists x\varphi \to \varphi\frac{c}{x}) \in W(S_n) \subseteq \Psi$ for the constant $c := c_{\exists x\varphi} \in S_{n+1}$. $\square$

Lemma 33 (E&F V.3.5: Zorn's Lemma)**.** Let M be a set and U a nonempty set of subsets of M. A *chain in U* is a nonempty $\mathcal{V} \subseteq U$ such that $V_1 \subseteq V_2$ or $V_2 \subseteq V_1$ for all $V_1, V_2 \in \mathcal{V}$. If

$$\bigcup_{V \in \mathcal{V}} V \;\in\; U \qquad \text{for every chain } \mathcal{V} \text{ in } U,$$

then U has a maximal element, i.e. some $U_0 \in U$ for which there is no $U_1 \in U$ with $U_0 \subsetneq U_1$.

Stated here in the form needed, for a family of sets ordered by inclusion, rather than for a general partial order. It is equivalent to the axiom of choice and is taken without proof; it replaces the enumeration of L^S that carried Lemma 30 in the countable case.

Lemma 34 (E&F V.3.2: negation completion, general case)**.** Let $\Psi \subseteq L^S$ with $\mathrm{Con}_S\,\Psi$. Then there is a consistent, negation complete Θ with $\Psi \subseteq \Theta \subseteq L^S$.

Proof. Put $U := \{\Phi \mid \Psi \subseteq \Phi \subseteq L^S,\ \mathrm{Con}_S\,\Phi\}$, nonempty as $\Psi \in U$, and take $M := L^S$. Let $\mathcal{V}$ be a chain in U and put $\Theta_1 := \bigcup_{\Phi \in \mathcal{V}} \Phi$. Then $\mathrm{Con}_S\,\Theta_1$: a finite $\Theta_0 = \{\varphi_1, \ldots, \varphi_n\} \subseteq \Theta_1$ has each $\varphi_i \in \Phi_i$ for some $\Phi_i \in \mathcal{V}$, and since $\mathcal{V}$ is a chain these may be numbered $\Phi_1 \subseteq \cdots \subseteq \Phi_n$, so $\Theta_0 \subseteq \Phi_n$ and $\mathrm{Con}_S\,\Theta_0$ by Lemma 20; now apply Lemma 22. So $\Theta_1 \in U$, and Lemma 33 yields a maximal $\Theta \in U$.

Θ is negation complete: for $\varphi \in L^S$, Lemma 24(c) gives $\mathrm{Con}_S\,\Theta \cup \{\varphi\}$ or $\mathrm{Con}_S\,\Theta \cup \{\neg\varphi\}$, so by maximality $\Theta = \Theta \cup \{\varphi\}$ or $\Theta = \Theta \cup \{\neg\varphi\}$, i.e. $\Theta \vdash \varphi$ or $\Theta \vdash \neg\varphi$. $\square$

Corollary 7 (E&F V.3.3)**.** Every consistent $\Phi \subseteq L^S$ is satisfiable, for arbitrary S.

Proof. As for Corollary 6, using Lemmas 32 and 34 in place of Lemmas 29 and 30, then Corollary 5. The passage to the larger symbol set S' is harmless, since satisfiability does not depend on the symbol set by Lemma 12. $\square$

4.4 The Completeness Theorem

Theorem 6 (E&F V.4.1: Completeness Theorem)**.** For $\Phi \subseteq L^S$ and $\varphi \in L^S$: if $\Phi \models \varphi$ then $\Phi \vdash_S \varphi$.

Proof. Corollary 7 is $(**)$, and $(*)$ follows from it by the argument at the start of the chapter. □

Theorem 7 (E&F V.4.2: Adequacy of $\mathcal{S}$)**.** (a) $\Phi \models \varphi$ iff $\Phi \vdash \varphi$. (b) $\mathrm{Sat}\,\Phi$ iff $\mathrm{Con}\,\Phi$.

Proof. (a) combines Theorem 6 with the Correctness Theorem 3. (b) combines Corollary 7 with Lemma 23. □

Satisfiability and consequence are independent of the symbol set (Lemma 12), so by Theorem 7 so are derivability and consistency. The subscripts in $\vdash_S$ and Con_S may therefore be dropped.

Gödel proved the Completeness Theorem in 1928; the proof above is Henkin's.