The Threshold for Random k-SAT

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Outline

Introduction

Definitions and the Threshold Conjecture

- **Random k-SAT Model:** Let $F_k(n,m)$ be a random k-CNF formula formed by selecting m clauses uniformly, independently, and with replacement from all possible $(2^k n^k)$ k-clauses on n variables.
- **w.h.p.:** A sequence of events $\mathcal{E}_n$ occurs *with high probability* (w.h.p.) if if $\lim_{n\to\infty} \mathbf{P}[\mathcal{E}_n] = 1$.
- A sequence of events $\mathcal{E}_n$ occurs **with uniformly positive probability** if $\liminf_{n\to\infty} \mathbf{P}[\mathcal{E}_n] > 0$.
- **Threshold Definitions:** For each $k \geq 2$, define the bounds:
  - $r_k \equiv \sup\{r : F_k(n, rn) \text{ is satisfiable w.h.p.}\}$
  - $r_k^* \equiv \inf\{r : F_k(n, rn) \text{ is unsatisfiable w.h.p.}\}$
- It is easy to see that $r_k \leq r_k^*$.
- **Satisfiability Threshold Conjecture:** Asserts that for all $k \geq 3$, the threshold is sharp. That is:

$$r_k = r_k^*$$

The Main Results

Theorem 1: As $k \to \infty$,

$$r_k = r_k^*(1 - o(1))$$

Theorem 2: $r_k \sim 2^k \ln 2$. Specifically, there exists a sequence $\delta_k \to 0$ such that for all $k \geq 3$:

$$r_k \geq 2^k \ln 2 - (k+1)\frac{\ln 2}{2} - 1 - \delta_k$$

The Upper Bound

Classical Result: $r_k^* \leq 2^k \ln 2$

Proof via the First Moment Method:

- Fix any specific truth assignment.
- A uniformly random k-clause is falsified only if all k of its literals mismatch our assignment. This happens with probability 2^{-k}.
- Thus, the probability our assignment satisfies a random clause is $(1 - 2^{-k})$.
- For a formula with rn independent clauses, the probability it satisfies all of them is $(1 - 2^{-k})^{rn}$.
- Since there are 2^n possible truth assignments, the expected number of satisfying assignments, $E[X]$, is:

$$E[X] = 2^n(1 - 2^{-k})^{rn} = \left[2(1 - 2^{-k})^r\right]^n$$

- For $r \geq 2^k \ln 2$, the base of the exponent is strictly less than 1.
- Therefore, $E[X] = o(1)$. By Markov's inequality, $P(X > 0) \to 0$ w.h.p. □

The Lower Bound

Shifting the Goal

Theorem 3 (Friedgut (1999)): For each $k \geq 2$, there exists a sequence $r_k(n)$ such that for every $\epsilon > 0$:

$$\lim_{n\to\infty} \mathbf{P}[F_k(n, rn) \text{ is satisfiable}] = \begin{cases} 1 & \text{if } r = (1-\epsilon)r_k(n) \\ 0 & \text{if } r = (1+\epsilon)r_k(n) \end{cases}$$

Corollary 1: Fix $k \geq 2$. If $F_k(n, rn)$ is satisfiable with uniformly positive probability, then $r_k \geq r$. *Key Proof Steps:*

1. Suppose $r > r_k$, then there exists some $\epsilon > 0$ and a subsequence of n where $r \geq (1+\epsilon)r_k(n)$.
2. Satisfiability is strictly decreasing with density. By Theorem 3, $\mathbf{P}[\text{sat at } (1+\epsilon)r_k(n)] \to 0$. Monotonicity squeezes our probability along this subsequence to 0 as well.
3. If a subsequence limits to 0, the overall $\liminf$ must be 0. This contradicts our premise, proving $r \leq r_k$. □

New Goal: To establish the lower bound r_k, we only need to prove satisfiability occurs with a **uniformly positive probability**.

The Vanilla Method

Let X be the number of satisfying assignments. We need to look at $E[X^2]$ and $E[X]^2$ to apply the second moment method. For a k-CNF formula (F) with *independent* clauses $c_1, \ldots, c_m$:

$$E[X^2] = E\left[\left(\sum_{\sigma} \mathbf{1}_{\sigma \text{ sat F}}\right)^2\right] = \sum_{\sigma,\tau} \prod_{c_i} E[\mathbf{1}_{\sigma,\tau \text{ sat } c_i}]$$

$E[\mathbf{1}_{\sigma,\tau \text{ sat } c_i}]$ is the probability that *both* σ and τ satisfy a single random clause. This probability depends on their **overlap**, $z = \alpha n$ (the number of variables where σ and τ agree):

$$P[\sigma, \tau \text{ sat } c_i] = 1 - 2^{1-k} + 2^{-k}\alpha^k \equiv f_S(\alpha)$$

Observe that $f_S(1/2) = \left(1 - 2^{-k}\right)^2 = P[\sigma \text{ sat } c_i]^2$.

The Vanilla Method

The number of pairs that share z variables is exactly $2^n\binom{n}{z}$. Hence:

$$E[X^2] = 2^n \sum_{z=0}^{n} \binom{n}{z} f_S(z/n)^m$$

From our observation in the last slide:

$$E[X]^2 = 2^{2n}(1-2^{-k})^{2m} = 2^{2n} f_S(1/2)^m$$

What do we WANT to happen?

We want $P(X > 0) \geq E[X]^2 / E[X^2]$ to be positive as $n \to \infty$. This means we want $E[X^2]$ to be at most a constant multiple of $E[X]^2$.

The Vanilla Method: The Failure

Using Stirling's approximation, $\binom{n}{\alpha n} \approx (\alpha^\alpha(1-\alpha)^{1-\alpha})^{-n} \times \text{poly (n)}$.
Bounding the sum by its maximum term yields:

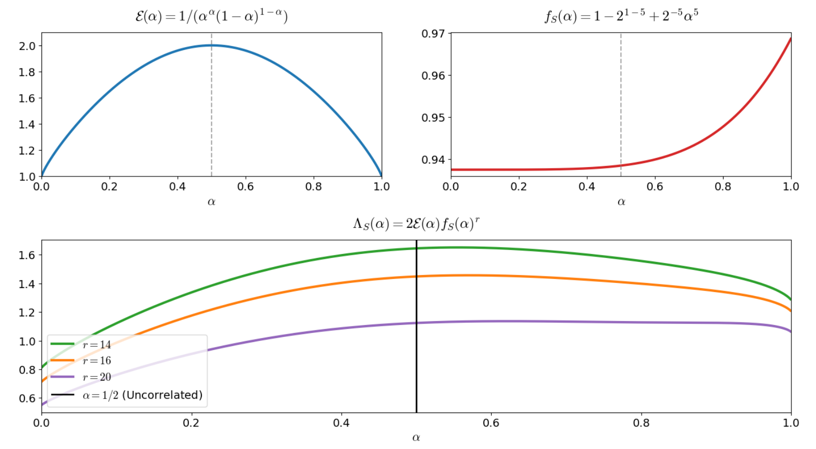
$$E[X^2] \geq \left(\max_{0\leq\alpha\leq 1} 2\left[\frac{f_S(\alpha)^r}{\alpha^\alpha(1-\alpha)^{1-\alpha}}\right]\right)^n \times \text{poly}(n)$$

Let $\Lambda_S(\alpha) \equiv 2\frac{f_S(\alpha)^r}{\alpha^\alpha(1-\alpha)^{1-\alpha}}$. Notice that $\Lambda_S(1/2)^n = E[X]^2$.

Why the method fails:

- We need the global maximum of $\Lambda_S(\alpha)$ to be exactly at $\alpha = 1/2$.
- At $\alpha = 1/2$, the denominator $\alpha^\alpha(1-\alpha)^{1-\alpha}$ is minimized, so its derivative is 0.
- However, $f_S(\alpha) = 1 - 2^{1-k} + 2^{-k}\alpha^k$ is strictly increasing in $(0, 1)$, so its derivative is positive at $1/2$.
- Thus, $\Lambda_S'(1/2) > 0$. The true maximum occurs at some $\alpha > 1/2$.
- $E[X^2]$ grows exponentially faster than $E[X]^2$, driving the probability bound to 0.

The figure below illustrates the behavior of $\Lambda_S(\alpha)$:



The Intuition

Why did the Vanilla Method fail? The failure is caused by **populism**. For a random k-SAT, assignments that satisfy many literals is likely to satisfy the clauses. These types of assignments are highly correlated. This correlation pushes the peak of our sum to $\alpha > 1/2$, and blows up the variance.

The Solution: "Balanced" Assignments

- If we shift our focus to "middle of the road" assignments (which satisfy roughly half the literals), we avoid the high correlation.
- introduce a **weighting scheme**. By assigning a weight to each satisfying assignment based on the number of literals it satisfies, we can penalize the populist assignments and artificially restore the peak at $\alpha = 1/2$.

Weighted Second Moment

Weighted Second Moments

The Second Moment Method works for *any* random variable X as long as $X > 0$ guarantees the formula is satisfiable.

We can redefine X as a weighted sum over all assignments:

$$X = \sum_{\sigma} w(\sigma, F)$$

This is valid as long as $w(\sigma, F) = 0$ whenever σ falsifies the formula F.

To compute expectations, we restrict ourselves to weights that factorize over the independent clauses: $w(\sigma, F) = \prod_c w(\sigma, c)$.

By linearity of expectation, clause independence:

$$\mathbf{E}[X] = \sum_{\sigma} \prod_{c} \mathbf{E}[w(\sigma, c)] = 2^n \left(\mathbf{E}[w(\sigma, c)]\right)^m$$

$$\mathbf{E}[X^2] = \sum_{\sigma,\tau} \prod_{c} \mathbf{E}[w(\sigma, c) w(\tau, c)] = \sum_{\sigma,\tau} \left(\mathbf{E}[w(\sigma, c) w(\tau, c)]\right)^m$$

Weighted Second Moments: Vector Mapping

Let $\mathbf{v} \in \{-1, +1\}^k$ represent how assignment σ evaluates the k literals of clause c (v_i is $+1$ if the literal is satisfied and -1 if it is falsified.)

We can write our weight function as $w(\sigma, c) = w(\mathbf{v})$. Crucially, $w(-1, \ldots, -1) = 0$ because falsified clauses cannot be counted. For a pair of assignments σ, τ with overlap α, we define their expected weighted correlation over a random clause as $f_w(\alpha)$:

$$f_w(\alpha) \equiv \mathbf{E}[w(\sigma, c)w(\tau, c)] = \sum_{u,v \in A} w(u)w(v)\, 2^{-k} \prod_{i=1}^{k} \left(\alpha^{1_{u_i = v_i}} (1-\alpha)^{1_{u_i \neq v_i}}\right)$$

This turns our second moment into the familiar sum:

$$\mathbf{E}[X^2] = 2^n \sum_{z=0}^{n} \binom{n}{z} f_w(z/n)^m \leq \left(\max_{0 \leq \alpha \leq 1} \Lambda_w(\alpha)\right)^n \times \text{poly}(n)$$

Observe that:

$$\mathbf{E}[X]^2 = 2^{2n} \left(\mathbf{E}[w(\sigma, c)]\right)^{2m} = 2^{2n} f_w(1/2)^m = \Lambda_w(1/2)^n$$

Center of Mass Rule

Let $\Phi_{\mathbf{u},\mathbf{v}}(\alpha) = 2^{-k} \prod_{i=1}^{k} \left(\alpha^{\mathbf{1}_{u_i=v_i}}(1-\alpha)^{\mathbf{1}_{u_i \neq v_i}}\right)$ and $\mathcal{E}(\alpha) = \left(\alpha^{\alpha}(1-\alpha)^{1-\alpha}\right)^{-1}$.

The polynomial factor is $O(1)$. This shows that if $\Lambda_w(1/2)$ is the global maximum, then $\mathbf{E}[X^2]/\mathbf{E}[X]^2 = O(1)$, allowing the second moment method to succeed.

A necessary condition for a global maximum is $\Lambda_w'(1/2) = 0$. Since $\Lambda_w(\alpha) = 2\mathcal{E}(\alpha) f_w(\alpha)^r$ and $\mathcal{E}'(1/2) = 0$, this requires $f_w'(1/2) = 0$.

$$f_w'(\alpha) = \sum_{\mathbf{u},\mathbf{v}} w(\mathbf{u}) w(\mathbf{v}) \Phi_{\mathbf{u},\mathbf{v}}(\alpha) \sum_{i=1}^{k} \left(\frac{\mathbf{1}_{u_i=v_i}}{\alpha} - \frac{\mathbf{1}_{u_i \neq v_i}}{1-\alpha} \right)$$

Evaluating at $\alpha = 1/2$, we have $\Phi_{\mathbf{u},\mathbf{v}}(1/2) = 2^{-2k}$ and the inner sum becomes $2(\mathbf{u} \cdot \mathbf{v})$:

$$2^{2k-1} f_w'(1/2) = \sum_{\mathbf{u},\mathbf{v}} w(\mathbf{u}) w(\mathbf{v}) (\mathbf{u} \cdot \mathbf{v}) = \left(\sum_{\mathbf{u}} w(\mathbf{u}) \mathbf{u} \right)^2$$

$$\therefore f_w'(1/2) = 0 \iff \sum w(\mathbf{v}) \mathbf{v} = 0$$

The Optimal Weighting Scheme

Our weight function w must satisfy two criteria:

1. $w(-1, \ldots, -1) = 0$ (Falsified clauses cannot be counted).
2. $\sum_{\mathbf{v}} w(\mathbf{v})\mathbf{v} = 0$ (The derivative at $1/2$ must be zero).

To push r as high as possible, we must keep $\mathbf{E}[X]$ large. This requires w to be as close to the uniform vanilla weights w_S as possible, which means **maximizing entropy** subject to our constraints. Let $|\mathbf{v}|$ denote the number of $+1$s in $\mathbf{v}$. The sum of the coordinates of $\mathbf{v}$ is exactly $2|\mathbf{v}| - k$. We obtain a necessary scalar condition:

$$\sum_{\mathbf{v} \neq (-1,\ldots,-1)} w(\mathbf{v})(2|\mathbf{v}| - k) = 0$$

Maximizing entropy subject to this constraint is a standard Lagrange multipliers problem. The unique solution is:

$$w(\mathbf{v}) = \frac{1}{Z} \lambda^{|\mathbf{v}|} \mathbf{1}_{\mathbf{v} \neq (-1,\ldots,-1)}$$

where Z is a normalizing constant and λ is chosen to satisfy the constraints.

Back to the Lower Bound

Theorem 4

Theorem 4. There exists a sequence $\beta_k \to 0$ such that for all $k \geq 3$,

$$r_k \geq 2^k \ln 2 - 2(k+1) \ln 2 - 1 - \beta_k$$

To prove this, we formalize our penalty weighting scheme. Let $H(\sigma, F)$ be the number of satisfied literal occurrences minus the number of unsatisfied ones.

Our new random variable is the sum of these exponentially weighted assignments:

$$X = \sum_{\sigma} \gamma^{H(\sigma,F)} \mathbf{1}_{\sigma \in \mathcal{S}(F)}$$

Note: $H(\mathbf{v}) = 2|\mathbf{v}| - k$ *for a single clause. So,* γ *works as* $\gamma = \sqrt{\lambda}$*.*

Additional Note: $\mathcal{S}(F)$ *is the set of satisfying assignments for formula* F*.*

The First Moment

For a single random clause $c = l_1 \vee \cdots \vee l_k$:

$$\mathbf{E}[\gamma^{H(\sigma,c)} \mathbf{1}_{\sigma \in \mathcal{S}(c)}] = \mathbf{E}[\gamma^{H(\sigma,c)}] - \mathbf{E}[\gamma^{-k} \mathbf{1}_{\sigma \notin \mathcal{S}(c)}]$$

k literals in a clause are chosen independently and uniformly at random:

$$\mathbf{E}[\gamma^{H(\sigma,c)}] = \prod_{i=1}^{k} \mathbf{E}[\gamma^{H(\sigma,\ell_i)}] = \left(\frac{\gamma + \gamma^{-1}}{2}\right)^k$$

The probability of falsifying all k literals is 2^{-k}, giving the second term $(2\gamma)^{-k}$. Defining this difference as $\psi(\gamma)$:

$$\psi(\gamma) \equiv \left(\frac{\gamma + \gamma^{-1}}{2}\right)^k - (2\gamma)^{-k}$$

Thus, over all 2^n assignments and rn independent clauses:

$$\mathbf{E}[X] = \sum_{\sigma} \mathbb{E}\left[\prod_{c_i} \gamma^{H(\sigma,c_i)} \, 1_{\sigma \in S(c_i)}\right] = (2\psi(\gamma)^r)^n$$

The Second Moment

Let σ, τ have overlap $z = \alpha n$. To find the expected correlation over a single clause, we use Inclusion-Exclusion on the indicator variable:

$$\mathbf{1}_{\sigma,\tau \in \mathcal{S}(c)} = 1 - \mathbf{1}_{\sigma \notin \mathcal{S}(c)} - \mathbf{1}_{\tau \notin \mathcal{S}(c)} + \mathbf{1}_{\sigma,\tau \notin \mathcal{S}(c)}$$

$$\text{Term 1: } \mathbf{E}[\gamma^{H(\sigma,\ell_i)+H(\tau,\ell_i)}] = \alpha \left(\frac{\gamma^2 + \gamma^{-2}}{2} \right) + 1 - \alpha$$

If σ falsifies the clause, every literal in σ is "wrong" (prob 2^{-k}). The weight contributed by τ depends on whether it agrees with σ (prob α) or disagrees (prob $1 - \alpha$):

$$\text{Term 2 (and 3): } \mathbb{E}\left[\gamma^{H(\sigma,\ell_i)+H(\tau,\ell_i)}\, 1_{\sigma \notin S(c)}\right] = 2^{-k} \left(\alpha\gamma^{-2} + (1-\alpha)\right)$$

If both falsify the clause, they must perfectly agree on all "wrong" literals:

$$\text{Term 4: } \mathbb{E}\left[\gamma^{H(\sigma,\ell_i)+H(\tau,\ell_i)}\, 1_{\sigma,\tau \notin S(c)}\right] = 2^{-k}\alpha^k \left(\gamma^{-2}\right)$$

The Second Moment

Combining the terms gives the full expectation for one clause:

$$\mathbf{E}[\gamma^{H(\sigma,c)+H(\tau,c)}\mathbf{1}_{\sigma,\tau\in\mathcal{S}(c)}] =$$

$$\left(\alpha\frac{\gamma^2+\gamma^{-2}}{2}+1-\alpha\right)^k - 2^{1-k}(\alpha\gamma^{-2}+1-\alpha)^k + 2^{-k}(\alpha\gamma^{-2})^k$$

To clean this up, we make the substitution $\gamma^2 = 1-\varepsilon$. Pulling out a common denominator of $2^k(1-\varepsilon)^k$, the numerator becomes a clean polynomial in α, which we define as $f(\alpha)$:

$$f(\alpha) \equiv (2-2\varepsilon+\alpha\varepsilon^2)^k - 2(1-\varepsilon+\alpha\varepsilon)^k + \alpha^k$$

Since the *rn* clauses are i.i.d., we multiply over all clauses and sum over all overlaps to get our final Second Moment:

$$\mathbf{E}[X^2] = 2^n \sum_{z=0}^{n} \binom{n}{z} \left(\frac{f(z/n)}{2^k(1-\varepsilon)^k}\right)^{rn}$$

Lemma 2

To bound our sum $\mathbf{E}[X^2]$, we rely on a standard analytical tool (Achlioptas and Moore 2002).

Lemma 2: For a sum $S_n = \sum_{z=0}^{n} \binom{n}{z} \phi(z/n)^n$, we isolate the exponential base:

$$g(\alpha) = \frac{\phi(\alpha)}{\alpha^\alpha (1-\alpha)^{1-\alpha}}$$

. If $g(\alpha)$ has a strict global maximum $g_{\max}$ at some $\alpha_{\max} \in (0,1)$, and $g''(\alpha_{\max}) < 0$, then the sum is bounded as follows for all sufficiently large n:

$$B \times g_{\max}^n \leq S_n \leq C \times g_{\max}^n$$

Insight: The $\Theta(n^{1/2})$ spread around the peak exactly cancels the polynomial decay from Stirling's approximation, leaving an $O(1)$ constant factor!

Lemma 3

We apply Lemma 2 to our second moment by defining our specific function:

$$g_r(\alpha) = \frac{f(\alpha)^r}{\alpha^\alpha(1-\alpha)^{1-\alpha}}$$

Lemma 3: Let our penalty parameter ε be defined by $\varepsilon(2-\varepsilon)^{k-1} = 1$. Let our clause density bound be $s_k = 2^k \log 2 - 2\log 2(k+1) - 1 - 3/k$.

If $k \geq 22$ and $r \leq s_k$, then $g_r(\alpha)$ has a strict global maximum at $\alpha = 1/2$, and $g_r''(1/2) < 0$.

Note: This equation for ε guarantees that the peak stays perfectly at 1/2. The density bound s_k guarantees that the secondary peak (which naturally forms near $\alpha \to 1$ due to populism) stays lower than our forced peak at 1/2.

Concluding Theorem 4

By combining Lemmas 2 and 3, we successfully bound the second moment:

$$\mathbf{E}[X^2] < C \times \left(\frac{2g_r(1/2)}{(2(1-\varepsilon))^k} \right)^n$$

A direct algebraic evaluation of the First Moment ($\mathbf{E}[X]^2$) yields the exact same exponential base:

$$\mathbf{E}[X]^2 = (2\psi(\gamma)^r)^{2n} = \cdots = \left(\frac{2g_r(1/2)}{(2(1-\varepsilon))^k} \right)^n$$

Substituting this, the n-dependent terms vanish, yielding:

$$\mathbf{E}[X^2] < C \times \mathbf{E}[X]^2$$

Therefore, $\mathbf{P}[X > 0] > 1/C > 0$. By Friedgut's sharp threshold (Theorem 3), success with positive probability implies success *with high probability*. Theorem 4 is proved!

Proving the Lemma

The ε Condition

Recall our scalar condition necessary for $f'_w(1/2) = 0$:

$$\sum_{\mathbf{v} \neq (-1,\ldots,-1)} w(\mathbf{v})(2|\mathbf{v}| - k) = 0$$

Substituting our optimal weights $w(\mathbf{v}) \propto \lambda^{|\mathbf{v}|}$, we group by the number of satisfied literals $j = |\mathbf{v}|$:

$$\sum_{j=1}^{k} \binom{k}{j} \lambda^j (2j - k) = 0$$

Through algebraic simplification, this elegantly reduces to a single constraint on our penalty parameter λ:

$$(1 + \lambda)^{k-1}(1 - \lambda) = 1$$

In the previous section, we defined $\gamma^2 = \lambda$ and made the substitution $\gamma^2 = 1 - \varepsilon$. Substituting $\lambda = 1 - \varepsilon$ directly into the constraint yields:

$$(2 - \varepsilon)^{k-1}\varepsilon = 1$$

Lemmas 4 & 7

We must prove $g_r(\alpha)$ has its strict global maximum exactly at $1/2$. We use a "divide and conquer" strategy.

Lemma 4: Right-Side Dominance

- We evaluate $f(1/2 + x) - f(1/2 - x)$ using binomial expansion.
- By pairing symmetric terms, we prove the difference is positive for $x > 0$.
- *Result:* $\forall \epsilon, x > 0, g_r(1/2 + x) > g_r(1/2 - x)$.

Lemma 7: Bounding the Penalty

- By analyzing the roots of $q(x) = x(2 - x)^{k-1} = 1$, we tightly bound our parameter for $k \geq 3$: $\varepsilon \approx 2^{1-k}$.
- *Result:* The penalty ε is exponentially small. This allows us to make algebraic approximations in the later bounds.

Lemmas 5 & 6

We now break the right-hand interval into two distinct battlegrounds.

Lemma 5: The Descent $(1/2, 4/5]$

- **Preconditions:** $k \geq 22$ and $r \leq 2^k \log 2$.
- By our choice of ε, we mathematically guaranteed $g_r'(1/2) = 0$.
- We prove the first derivative $g_r'(\alpha) < 0$ for all $\alpha \in (1/2, 4/5]$. Thus, the function is strictly decreasing away from the peak.
- We evaluate the second derivative exactly at the peak to prove $g_r''(1/2) < 0$. This satisfies one of the conditions for Lemma 2.

Lemma 6

We must prove $g_r(1/2) > g_r(\alpha)$ for $\alpha \in (4/5, 1]$. By expanding the definition of g_r, this is algebraically equivalent to:

$$\left(\frac{f(\alpha)}{f(1/2)}\right)^r < 2\alpha^\alpha(1-\alpha)^{1-\alpha}$$

Let $h(\alpha) = -\alpha \log \alpha - (1-\alpha)\log(1-\alpha)$. Taking the logarithm of both sides and using the bound $\log(1+x) \le x$, we can isolate r:

$$r \le \frac{\log 2 - h(\alpha)}{f(\alpha) - f(1/2)} \times f(1/2)$$

By bounding the maximum possible growth of the correlation function, we prove $f(\alpha) - f(1/2) \le \alpha^k$. Substituting this denominator yields the ceiling r must not exceed:

$$r \le \phi(\alpha) \times f(1/2) \quad \text{where} \quad \phi(\alpha) \equiv \frac{\log 2 - h(\alpha)}{\alpha^k}$$

Lemma 6

To guarantee $r \le \phi(\alpha) f(1/2)$ holds for all $\alpha \in (4/5, 1]$, r must be less than the absolute minimum of the right-hand side.

Bounding $f(1/2)$ using Lemma 7
We need an explicit algebraic value for $f(1/2)$, which depends on ε. By substituting the bounds for our penalty ε established in Lemma 7, we obtain a rigorous lower bound:

$$f(1/2) > 2^k - 2k - 2 - 3k^2 2^{-k}$$

Bounding $\phi(\alpha)$
Calculus shows $\phi(\alpha) > \log 2 - 2^{-k} - \frac{2}{k^2 2^k}$.

Conclusion: Multiplying the worst-case minimum of $\phi(\alpha)$ by our explicit lower bound for $f(1/2)$ yields exactly s_k. Thus, the secondary peak is defeated if and only if $r \le s_k$.

References

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